Elliptic Functions of Jacobi Type *

What to do in 3D-XplorMath at the end of this text.

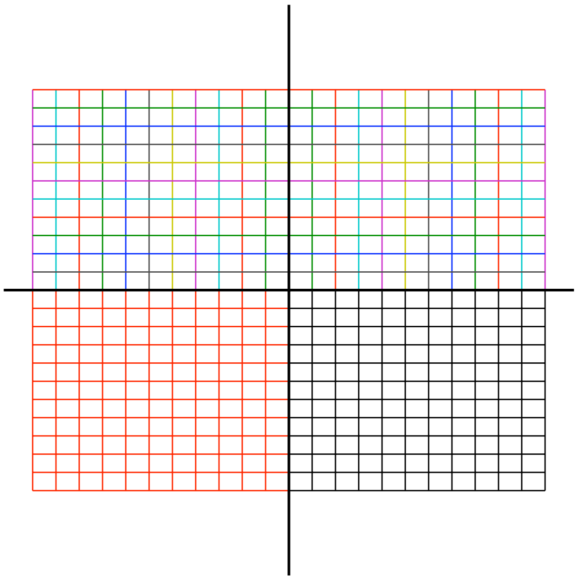
Elliptic functions are doubly periodic functions in the complex plane. A period of a function f is a number $\omega \in \mathbb{C}$ such that $f(z) = f(z+\omega)$ for all $z \in \mathbb{C}$. *Doubly periodic* means that the function has two periods ω_1, ω_2 with $\omega_1/\omega_2 \notin \mathbb{R}$. The set of all period translations is a lattice Γ, and Γ has some parallelogram as fundamental domain. Period translations identify parallel edges of this parallelogram to a torus and elliptic functions can therefore be viewed as functions on such a torus or equivalently as conformal maps from the torus to the Riemann sphere.

The simplest such functions are two-to-one maps from the torus to the sphere. They have either *one double pole* and are not very different from the Weierstrass $\wp$-function: $f(z) = a \cdot \wp(z) + b$, or they have *two simple poles;* the oldest of these are Jacobi's functions sn, cn, dn, more below.

These doubly periodic functions have various properties in common with the singly periodic trigonometric functions. They are inverse functions of certain integrals and therefore are solutions of first order ODEs, just like the inverse function of $\int_0^z 1/\sqrt{1-\zeta^2}d\zeta$ is a solution of the non-Lipschitz ODE $f'(z)^2 = 1 - f(z)^2$. And, as in this trigonometric case, differentiation of the first order ODE gives a second order ODE which is Lipschitz, here $f''(z) = -f(z)$. The trigonometric functions have more symmetries than

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

their translations. These symmetries give all values from the values on $[0, \pi/2] \times \mathbf{i} \cdot \mathbb{R}$. For the two-to-one elliptic functions all values occur already on half a torus and the further symmetries compute all these values from the values on one eighth of the torus. If the torus happens to be rectangular, it has further symmetries: reflections which are anticonformal. In this case, from the values on one sixteenth of the torus, one can obtain all other values via Möbius transformations. And, one can define interesting elliptic functions with the Riemann mapping theorem:
The rectangle in the first quadrant of the picture below is one sixteenth of a rectangular torus. The Riemann mapping theorem is used to map this rectangle to the quarter unit circle in the first quadrant of the second picture. Riemann's theorem allows to specify that three corners of the rectangle go to $1, 0, \mathbf{i}$ and at the 4th corner the derivative vanishes and two edges are mapped to the quarter circle:



Torus $\frac{1}{4}$-fundamental domain

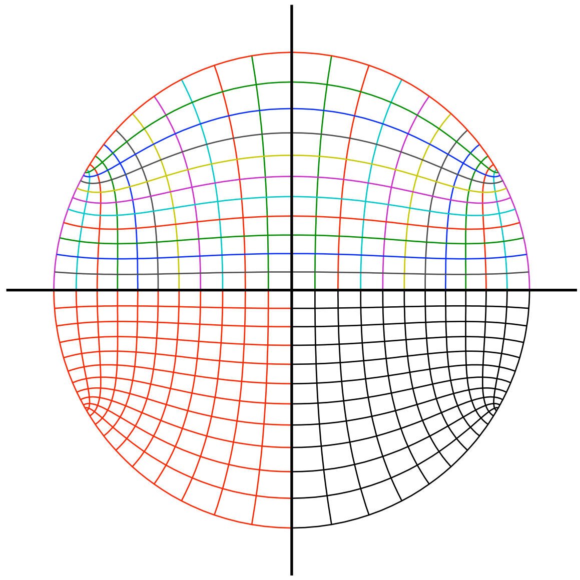


Image of J_D-function

The extension of this definition to a two-to-one map from the torus to the (Riemann) sphere is made possible by *Schwarz Reflection*. Its simplest version, for complex power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbb{R}$, says $f(\bar{z}) = \overline{f(z)}$. We use the next step: Instead of reflection in $\mathbb{R}$, reflection in any straight or circular boundary arc (in domain and range) extends the definition of the function.

While all rational functions can be obtained by rational operations from the **single** polynomial $P(z) = z$, one needs **two** elliptic functions on a given torus to obtain all others by rational operations from these. Two choices for such a second function are obtained if one maps three other vertices of the rectangle in the first quadrant to $0, 1, \mathbf{i}$ on the quarter circle in the first quadrant (but always origin to origin). In each case the derivative of the map vanishes at the last corner and the $90°$ angle between adjacent edges is opened to the $180°$ angle between the image arcs:

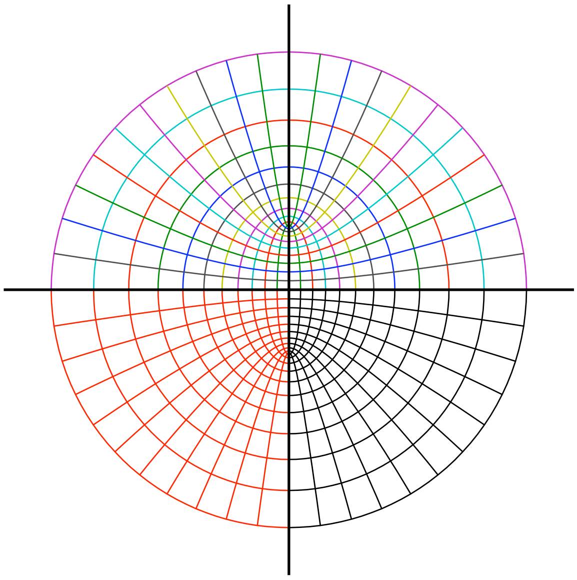


Image of J_E-function

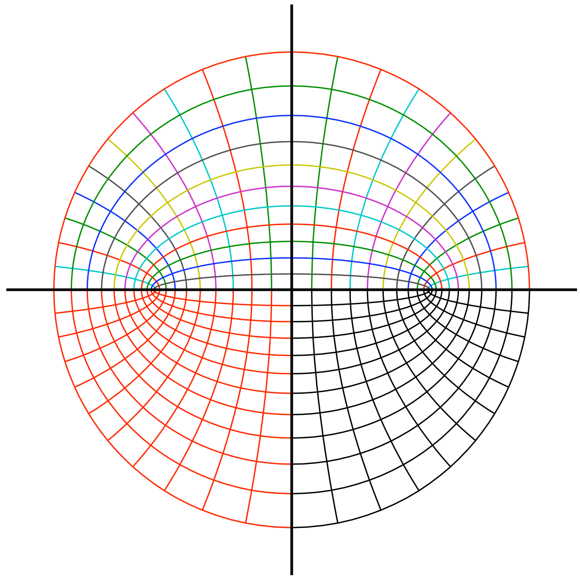


Image of J_F- or sn-function

The three functions J_D, J_E, J_F were developed in work on

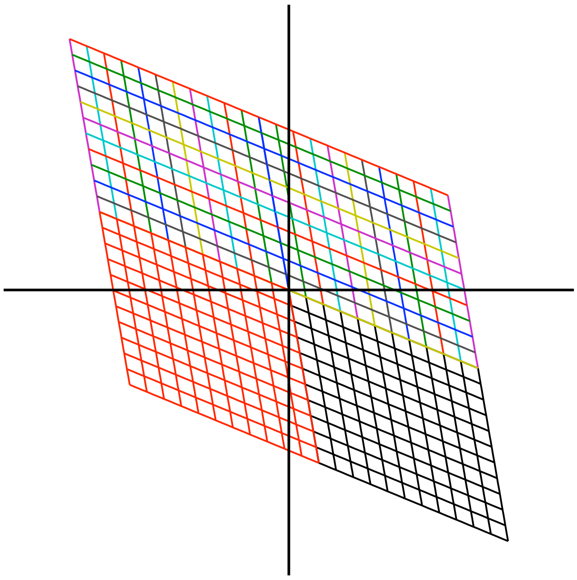
minimal surfaces. In this context they have two advantages over Jacobi's sn,cn,dn:
a) J_D, J_E, J_F are defined on the same torus while sn,cn,dn are defined on three different tori, which are closely related, namely doubly covered by a common rectangular torus. On this larger rectangular torus Jacobi's functions are of degree 4.
b) At points z_1, z_2 of the torus which are related by a symmetry of the four branch points of J_D, J_E or J_F, the **values** $J(z_1), J(z_2)$ are related by **isometries** of the Riemann sphere, while for sn, cn, dn the relations between the values are by more general Möbius transformations. In applications to minimal surfaces such isometric relations translate into symmetries of the minimal surface, while Möbius relations do not.
Example: On each rectangular torus we have Riemann's embedded minimal surface and its conjugate; the Gauss map of these surfaces is the geometrically normalized Weierstrass $\wp$-function (denoted $\wp_g$), not the original $\wp$-function. We have $\wp_g = J_E \cdot J_F$. If σ is a 180° rotation around a midpoint between the double zero and double pole of $\wp_g$, then $\wp_g(\sigma(z)) = -1/\wp_g(z)$.
Jacobi's sn-function and our J_F-function (on rectangular tori) are very closely related:
The branch values of sn are $\{\pm 1, \pm k\}$, modul $m = k^2$.
The branch values of J_F are $\{\pm F, \pm F^{-1}\}$, $F := \sqrt{k}$.
The fundamental domain of sn is such that $\mathrm{sn}'(0) = 1$.
The fundamental domain of J_F is such that $J_F'(0) = \frac{2}{F+1/F}$.

The function dn is also defined on a rectangular torus and has real branch values $\{\pm 1, \pm\sqrt{1-k^2}\}$.
The function cn is not defined on a rectangular torus.
$\mathrm{cn}(0), \mathrm{dn}(0) \neq 0,\ \mathrm{cn}'(0) = 0,\ \mathrm{dn}'(0) = 0.$

On non-rectangular tori we cannot define Jacobi type elliptic functions by the Riemann mapping theorem.
In *Symmetries of Elliptic Functions* we construct them with a more abstract tool: One can rotate any parallelogram torus by 180° around any of its points. This symmetry has four fixed points which are the vertices of a parallelogram with half the edgelength as the fundamental domain of the torus.

The quotient by such a symmetry is a conformal sphere!

The uniformization theorem of complex analysis states that every conformal sphere is biholomorphic to the Riemann sphere. Therefore we can make the quotient map into a function by specifying three points on the torus and call their images on the Riemann sphere $0, 1, \infty$. Example:



Torus $\frac{1}{4}$-fundamental domain

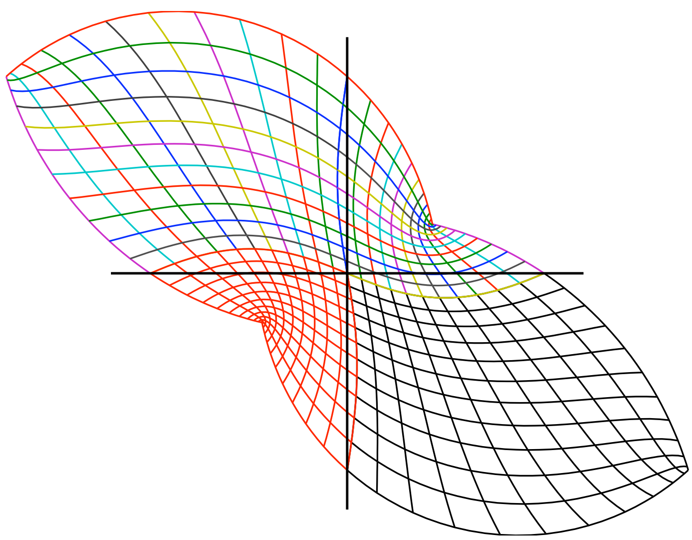


Image of J_D-function

180° rotation of the torus around the edge midpoints of the quarter domain and the corresponding 180° Möbius rotations extend the shown portion to a two-to-one conformal map from the torus to the sphere.
Our computation of these images uses the ODE. We scaled the function so that its branch values are $\{\pm B, \pm B^{-1}\}$. Our three Jacobi type functions J_D, J_E, J_F satisfy the ODE:

$$(J')^2 = J'(0)^2 \cdot \big(J^4 - (B^2 + B^{-2}) \cdot J^2 + 1\big).$$

The two functions on both sides of the equality sign agree because they have the same zeros and poles, hence are proportional, and $J'(0)^2$ is the correct proportionality factor. Differentiation of this ODE and cancellation of $2J'$ give the more harmless 2nd order nonlinear ODE (which is needed because Runge-Kutta cannot integrate the 1st order ODE in the vicinity of the zeros of the right side, called the branch values of J):

$$J'' = J'(0)^2 \cdot \big(2J^2 - (B^2 + B^{-2})\big) \cdot J.$$

These equations can be used for J_D, J_E, J_F. To be on the same torus one has to transform the branch values as:

$$E = \frac{D-1}{D+1},\ F = \mathbf{i} \cdot \frac{D-\mathbf{i}}{D+\mathbf{i}},\ F = -\frac{E-\mathbf{i}}{E+\mathbf{i}},$$

and to use the same scale for the fundamental domain:

$$J'_D(0) = 1,\ J'_E(0) = \frac{D - 1/D}{2\mathbf{i}},\ J'_F(0) = \frac{D + 1/D}{2}.$$

In 3D-XplorMath the `morphing parameter` for Jacobi's sn is the modul m and for J_D, J_E, J_F it is the branch value D of J_D in the 1st quadrant. Note that $|D| = 1$ for rectangular tori and Re $(D) =$ Im $(D) > 0$ for rhombic tori.
`Input` $\log(D)$ into dd, rectangular case: $dd \in \mathbf{i} \cdot (0, \pi/2)$.
Users cannot change the size of the domain of elliptic functions, it is always one half of the torus, chosen so that the values cover the Riemann Sphere once. In the domain we use a grid made up of eight copies of one sixteenth of the torus. The number of grid lines is the same in both directions so that the grid meshes are proportional to the fundamental domain. The default picture is, as always for our conformal maps, the image grid and the grid meshes show approximately the conformal type of the torus.
If one selects in the Action Menu `Show Image on Riemann Sphere` one can see the symmetries of these elliptic functions more clearly.
The entry `Show Inverse Function` in the Action Menu offers a second visualization. It assumes that standard polar coordinates on the Riemann Sphere are well known. The preimage of this polar grid is shown. Note that the preimages of all latitudes – except the equator – are pairs of congruent smooth closed curves. The preimage of the equator consists of four squares. The preimages of the northern and southern hemissphere are therefore easily recognized parts of the torus. – This visualization is completed only for rectangular tori.
H.K.