About the Unduloid

What are the different shapes that a soap film can take, or to put it somewhat differently, what can we say about the differential geometry of a mathematical surface that approximates a soap film?

An important physical characteristic of the soap film is its surface tension T. This depends only on the chemical composition of the liquid from which it is made, and so it is the same at each point of the film. The difference in air pressure between the two sides of the film is an environmental variable that is also clearly the same at all points of the film. Now it follows from physical principles (that we will take for granted here) that the mean curvature H of the soap film at any point is equal to P/T, and so we see that a soap film is always represented by a surface that has constant mean curvature.

For a soap film that we get by dipping a closed loop of wire into soapy water, the air pressure on both sides is clearly the same, so such a soap film must have mean curvature zero. Such surfaces are called *minimal surfaces*, since it can be shown that if we draw any small closed curve on the surface, the area of the part of the surface inside the curve is less than or equal to the area of any other surface bounded by the curve.

We consider minimal surfaces in considerable detail elsewhere, and here we shall be interested in the case of soap bubbles. These are soap films that (perhaps together with some other surfaces) enclose a bounded region of space (the "inside" of the bubble). For bubbles the pressure will be slightly greater on the inside than on the outside, so that the surface is what is called a *CMC surface*, that is it has **non-zero** constant mean curvature (and of course for the floating type it is often just a sphere).

If one blows a soap bubble between two parallel glass plates then one can obtain CMC surfaces that are surfaces of revolution, and such CMC surfaces are called *Unduloids*.

Consider a curve in the x-y-plane, given parametrically by $x = x(t), y = y(t)$, or as a graph $(x, f(x))$ of a function f. If one rotates this curve about the x-axis, it is easy to compute an expression for the mean curvature H of the resulting surface of revolution in terms of the first and second derivative of $x(t)$ and $y(t)$ (or, in the graph description, the derivatives of f). If this expression is set equal to a positive constant H, one gets differential equations for the functions $x(t)$ and $y(t)$ (respectively for the function f), and solving these ODE provides a method for finding all CMC surfaces of revolution. Delaunay studied this problem in 1841, and being an expert on the theory of roulettes (i.e., a locus traced out by a point attached to curve as that curve rolls on a line), he recognized that the solutions of this differential equation could be identified with the roulettes traced out by a focus of a conic section as it rolls along the x-axis. The special case that the conic is an ellipse gives the Unduloid. In 3D-XplorMath, the Unduloid is literally constructed by this double process of first rolling an ellipse and tracking one of its foci and then rotating the resulting curve around the x-axis.

The default morph shows a family of unduloids that starts with a cylinder and deforms towards a chain of spheres. With the rolling construction of the Unduloid, we cannot reach the chain of spheres because the parameter lines become concentrated near the narrowing necks of the surfaces. However, if one resizes these necks so they have constant waist size, then the necks converge to (minimal) Catenoids. This fact was very important in the construction of very general examples by Kapouleas.

H.K.