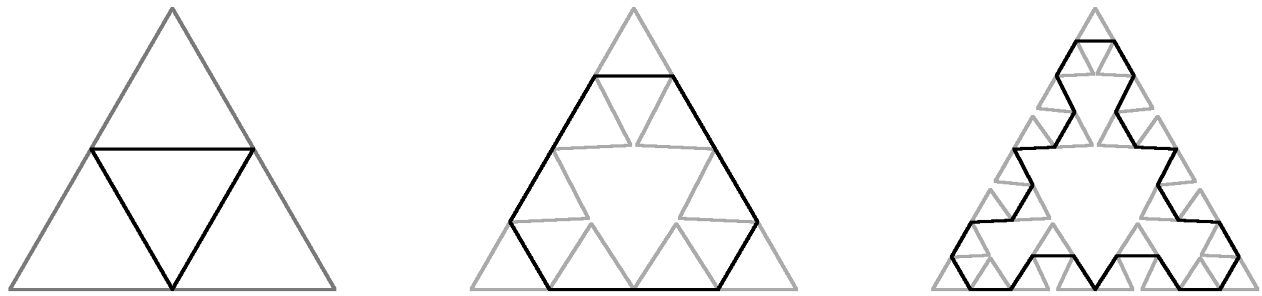
The Sierpinski Triangle, The Sierpinski Curve*

The Sierpinski Triangle is a well known example of a "large" compact *set* without interior points. It is defined by the following construction:
Start with an equilateral triangle and subdivide it into four congruent equilateral triangles. Remove the middle one. Subdivide the remaining triangles again and remove in each the middle one. Repeat this procedure. Each step reduces the area by a factor 3/4. – But more is true:

Sierpinski's Triangle is the image of a continuous curve.

As in the other fractal curves in 3DXM we have to define an iteratively defined and uniformly convergent sequence of polygonal curves. As in the case of the Hilbert square filling curve there is an easier construction by *non-injective* curves which, however, can be modified to give better looking *injective* approximations. In the following illustration we have chosen the 3DXM parameter $bb = 0.49$, because for $bb < 0.5$ the easier construction also gives injective approximations. ($bb = 0.5$ gives Sierpinski's curve.)



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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

The starting polygonal curve has the vertices and the edge midpoints of an equilateral triangle as its vertices. The initial point is the midpoint of the bottom edge. The curve that joins every second vertex of the starting curve is the triangle in the middle. We view the starting curve as passing through two edges of each of the three outer triangles. We only have to describe for one of these triangles how the next iteration is obtained. We will obtain curves that always run through two edges of each triangle, and the basic iteration can always be applied. If we join every second vertex of the resulting curves then we obtain the injective approximations of the Sierpinski Curve.
The basic iteration step, for one triangle:
First add the two midpoints of the traversed edges of the triangle. Two more points are added, one over the first and one over the last of the four subsegments. The points lie in the inside of the traversed triangle and they are the tips of isocele triangles whose base is the first, resp. the last, of the four subsegments. In the case of the Sierpinski Curve these isocele triangles are in fact equilateral. If the parameter bb is smaller than 0.5 then the height of the isocele triangle is by the factor $bb/0.5$ smaller than the height of the equilateral triangle – thus avoiding the creation of double points of the approximation.
The iterated polygonal curve joins the initial point of the first edge to the first tip, continues to the first edge-midpoint, passes through the vertex of the original triangle to the second edge-midpoint, continues through the second tip

and ends at the final point of the last segment. The iterated polygonal curve traverses three triangles, two edges in each. Therefore the iteration step can be repeated.
The default `Morph` from the Animation Menu of 3DXM varies bb from 1/3 to 1/2 thus joining the first triangle contour by a family of continuous (and injective) curves to the Sierpinski Curve.
Finally, one can choose in the Action Menu to map any selected Fractal curve by either the complex map $z \to z^2$ or by the complex exponential. The program waits for a mouse click and then chooses the mouse point as origin.

H.K.