About the Lopez-Ros No-Go Theorem

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The Lopez-Ros Theorem [LR] says that a complete, minimal embedding of a punctured sphere is either a Catenoid or a plane. Our example is parametrized by a 3-punctured sphere. Its Gauss map is $\mathrm{Gauss}(z) := cc(z-1)(z+1)$. The differential $dh = (z^2-1)/(z^2-ee)^2$ puts the punctures at $+ee$, $-ee$ and ∞. Parameter lines on the sphere extend polar coordinates around the punctures at $z = +ee$, $z = -ee$, $z = \infty$ in order to make the ends look nice.

A necessary condition for embeddedness is that the normals of all ends are parallel, i.e., $ee = 1$. In this case a residue computation shows that the period cannot be closed, in agreement with the theorem of Lopez-Ros. If $ee > 1$, then cc can be chosen to close the period, but then the catenoid ends are tilted so that they intersect the third (planar) end. The default morph in 3DXM shows what happens when ee approaches 1 while the

periods are always closed (with a closing value of cc that grows to infinity). In a properly scaled picture the surface looks more and more like two catenoids at larger and larger distance.
Since we also want to show morphs were the period opens up, the program uses cc as follows: If the user sets $cc = 0$ then the program recomputes cc to the period-closing value cc_{close}. Otherwise we only restrict cc by resetting it as

$$cc = \max(0.5 * cc_{close}, \min(cc, 2 * cc_{close})).$$

To choose one's own morph, note that the surface has a gap if cc is larger than cc_{close}, and that it intersects itself if cc is smaller than cc_{close}. As an example, compute first with $ee = 1.01, cc = 0$; then, in the Set Morphing Dialog, click the button *Initialize to current parameters*, and finally morph ee from the current value $ee0 = 1.01$ to some larger value, e.g. $ee1 = 1.08$.

References

[LR] F.J. Lopez and A. Ros, On embedded complete minimal surfaces of genus zero, Journal of Differential Geometry 33 (1), 1991, pp 293–300.

For a discussion of techniques for creating minimal surfaces with various qualitative features by appropriate choices of Weierstrass data, see either [KWH], or pages 192–217 of [DHKW].

[KWH] H. Karcher, F. Wei, and D. Hoffman, The genus one helicoid, and the minimal surfaces that led to its discovery, in "Global Analysis in Modern Mathematics, A Symposium in Honor of Richard Palais' Sixtieth Birthday", K. Uhlenbeck Editor, Publish or Perish Press, 1993

[DHKW] U. Dierkes, S. Hildebrand, A. Kuster, and O. Wohlrab, Minimal Surfaces I, Grundlehren der math. Wiss. v. 295 Springer-Verlag, 1991