The Logarithmic Spiral *

The parametric equations for the Logarithmic Spiral are:

$$x(t) = aa \cdot \exp(bb \cdot t) \cdot \cos(t)$$
$$y(t) = aa \cdot \exp(bb \cdot t) \cdot \sin(t).$$

This spiral is connected with the complex exponential as follows:

$$x(t) + i\,y(t) = aa\,\exp((bb + i)t).$$

The animation that is automatically displayed when you select Logarithmic Spiral from the Plane Curves menu shows the osculating circles of the spiral. Their midpoints draw another curve, the *evolute* of this spiral. These osculating circles illustrate an interesting theorem, namely if the curvature is a monotone function along a segment of a plane curve, then the osculating circles are nested - because the distance of the midpoints of two osculating circles is (by definition) the length of a secant of the evolute while the difference of their radii is the arc length of the evolute between the two midpoints. (See page 31 of J.J. Stoker's "Differential Geometry", Wiley-Interscience, 1969).

For the logarithmic spiral this implies that through every point of the plane minus the origin passes exactly one osculating circle. Étienne Ghys pointed out that this leads

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

to a surprise: The unit tangent vectors of the osculating circles define a vector field X on $\mathbb{R}^2 \setminus \{0\}$ – but this vector field has more integral curves, i.e. solution curves of the ODE $c'(t) = X(c(t))$, than just the osculating circles, namely also the logarithmic spiral. *How is this compatible with the uniqueness results of ODE solutions?* Read words backwards for explanation:

eht dleifrotcev si ton ztihcspiL gnola eht evruc.

Remarks about Spirals

All spirals with names share the following properties:
If they are left rotating their curvature is positive, and negative for right rotating ones. If they are outgoing, the absolute value of the curvature is monotone decreasing, and increasing for those which spiral inwards. Moreover, for every such curvature function $\kappa(s)$ one can find a spiral, with $\kappa(s)$ its curvature function, as follows:
First find an antiderivative $T(s)$ of $\kappa(s)$, i.e. $T'(s) = \kappa(s)$.
Next define a unit field $c'(s) := \big(\cos(T(s)),\ \sin(T(s))\big)$.
Finally obtain the spiral $c(s)$ by integrating $c'(s)$.

Therefore one may take the quoted curvature properties as definition of a spiral. This has the consequence that a spiral with curvature $\kappa_1(s)$ can be deformed into a spiral with curvature $\kappa_2(s)$ through spirals with curvature

$$\kappa_a(s) := (1-a) \cdot \kappa_1(s) + a \cdot \kappa_2(s).$$

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