Conic Sections, Kepler orbits

See also Parabola, Ellipse, Hyperbola and their ATOs.

For many properties of the conic sections a parametrization is not relevant. However, when Kepler discovered that planets and comets travel on conic sections around the sun then this discovery came with a companion: the speed on the orbit is such that angular momentum is preserved. In more elementary terms: the radial connection from the sun to the planet sweeps out equal areas in equal times. With the 3dfs demo we explain geometrically how this celestial parametrization is connected with the focal properties of conic sections. Here we give the *algebraic explanation* first.

An ellipse, parametrized as *affine image of a circle* and translated to the left is

$$P(\varphi) := (a\cos\varphi - e,\ b\sin\varphi).$$

If we choose $e := \sqrt{a^2 - b^2}$ then we have $|P(\varphi)| = (a - e\cos\varphi)$. This gives the connection with the oldest definition of an ellipse: The sum of the distances from $P(\varphi)$ to the two points $(\pm e, 0)$ is $2a$.

Next we compute the quantity A, equal to twice the area swept out by the position vector P, and also the derivatives of P and A:

$$A(\varphi) = \int_0^\varphi \det(P(\varphi), P'(\varphi))d\varphi,$$

$$P'(\varphi) = (-a\sin\varphi, b\cos\varphi),$$

$$A'(\varphi) = b(a - e\cos\varphi),$$

and we denote the function inverse to $A(\varphi)$ by $\Phi(A)$, so that,

$$\Phi(A(\varphi)) = \varphi, \qquad \Phi'(A) = \frac{1}{b(a - e\cos\Phi)}.$$

Let us write Q to denote the position when expressed as a function of A, i.e., $Q(A) := P(\Phi(A))$. Now Kepler's Second Law says that A proportional to time, or equivalently that A *is* the time in approriate units, so the velocity is $Q'(A) = P'(\Phi(A)) \cdot \Phi'(A)$, and the kinetic energy is:

$$
\begin{aligned}
K.E. = \frac{1}{2}Q'(A)^2 &= \frac{a^2\sin^2\varphi + b^2\sin^2\varphi}{2b^2(a - e\cos\varphi)^2} \\
&= \frac{a^2 - e^2\cos^2\varphi}{2b^2(a - e\cos\varphi)^2} \\
&= \frac{(a + e\cos\varphi)}{2b^2(a - e\cos\varphi)} \\
&= \frac{a}{b^2}\cdot\frac{1}{a - e\cos\varphi} - \frac{1}{2b^2}. \\
&= \frac{a}{b^2}\cdot\frac{1}{|P(\varphi)|} - \frac{1}{2b^2}.
\end{aligned}
$$

Thus, in units where we take twice the swept out area as the time, the potential energy can be read off by using the law of energy consevation, i.e., the fact that the kinetic energy plus the potential energy is constant. In fact, it follows from this that the potential energy at orbit point $Q(A(\varphi)) = P(\varphi)$ is equal to:

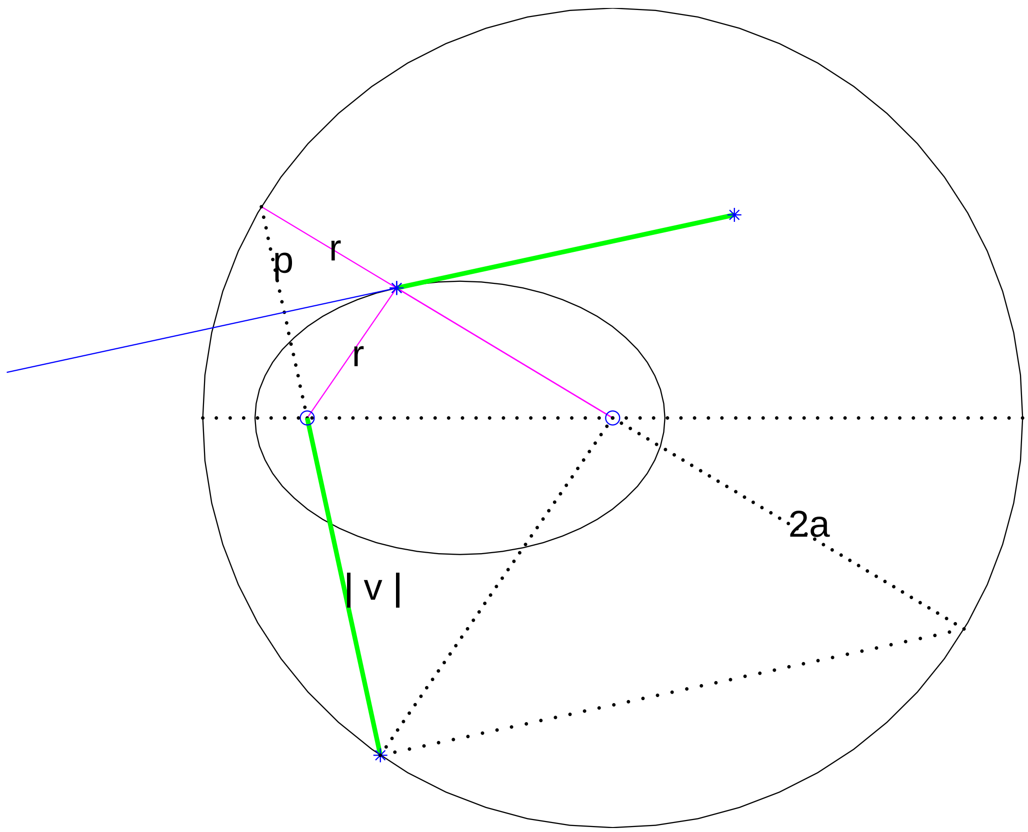
$$
-\frac{a}{b^2}\cdot\frac{1}{|P(\varphi)|},
$$

which is the famous $1/r$ law for the potential energy.

Now we present a *geometric proof.* The convincing simplicity of such arguments was also the motivation for Feynman's *Lost Lecture.* The starting point is the determination of the correct orbital speed by the property that the product of the speed $|v|$ with the distance p of the tangent line from the center is the constant angular momentum, Kepler's second law. Of course we can illustrate such a fact only if we also represent the size of velocities by the length of segments and we have to keep in mind that segments which illustrate a length and segments which illustrate a velocity are interpretated with different units.

Recall the following theorem about circles: if two secants of a circle intersect then the product of the subsegments of one secant ist the same as the product of the subsegments of the other secant.

This will be applied to the circle the radius of which is the length $2a$ of the major axis. (The midpoint is the other focus, not the sun.) The two secants intersect in the focus representing the sun: one secant is an extension of the major axis the other is perpendicular to the tangent line. The subsegments of the first secant have the lengths $2a-2e$ and $2a+2e$, where $2e$ is the distance between the foci. The subsegments of the second secant have one length $2p$ and one labeled $|v|$.



Kepler Ellipse with construction
of proper speed and potential.

The circle theorem says: $(2a-2e)\cdot(2a+2e) = 2p\cdot|v|$. Since the left side is constant we can interprete the segment labeled $|v|$ as representing the *correct orbital speed.*

Now that we know at each point of the orbit the correct speed we can deduce Newton's $1/r$-law for the gravitational potential, if we use *kinetic energy plus* *potential energy equals constant total energy.* In the illustration we have two similar right triangles, the small one has hypothenuse $= r$ and one other side $= p$, the big one has as hypothenuse a circle diameter of length $4a$ and the corresponding other side has length $2p + |v|$. Now we use the above $const :=$ $(2a - 2e) \cdot (2a + 2e) = 2p \cdot |v|$ to eliminate p from the proportion:

$$p : r = (2p + |v|) : 4a$$

This gives

$$2a/r = 1 + |v|/2p = 1 + v^2/const.$$

Up to physical constants (units) v^2 is the kinetic energy, so that (again up to units) $-1/r$ is the potential energy – since such a potential makes kinetic plus potential energy constant.

Another simple property of Kepler ellipses and hyperbolas is: Their velocity diagram, the so called *hodograph*, is a circle. Usually one simply translates the velocity vector from the orbit point to the sun. In our picture we see the velocity vector rotated by 90 degrees; indeed, it ends on the circle. This leads to a geometric representation of the *Runge-Lenz vector*:

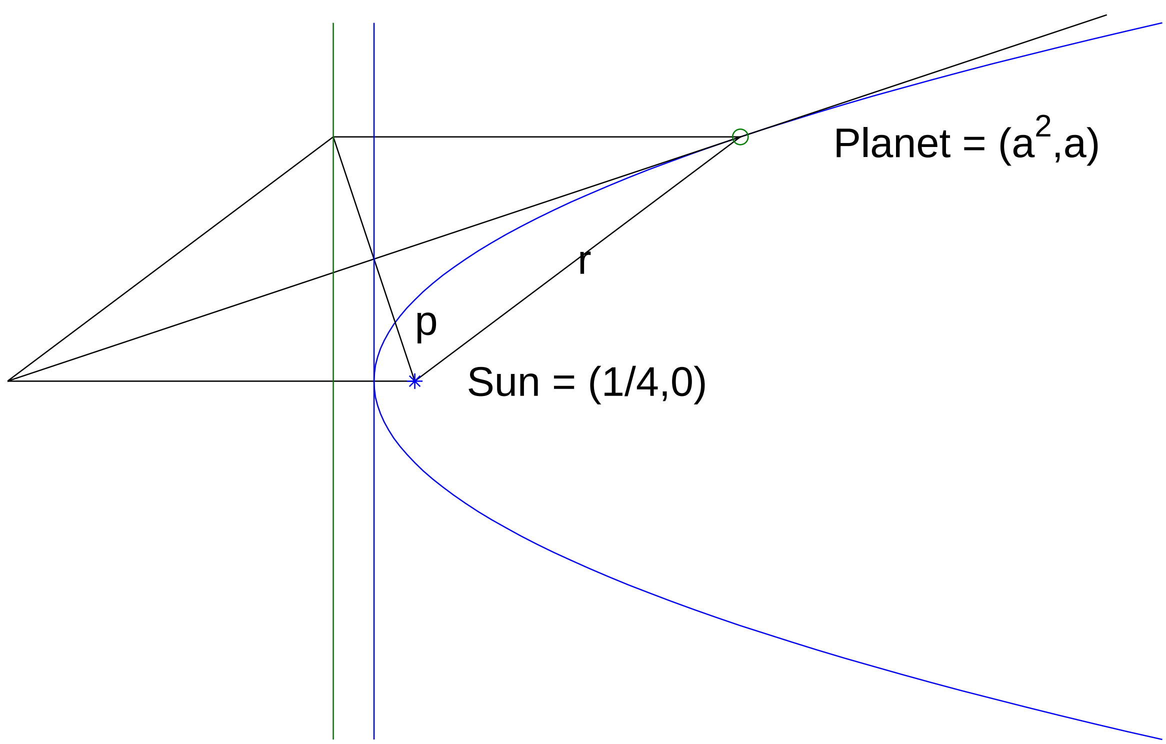
In our picture we really see the cross product of the (tangential) velocity vector with angular momentum (a constant vector orthogonal to the orbit plane). If we add to it a vector of constant length $2a$ and *parallel* to the position vector then we reach the midpoint of our circle, the other focal point of the orbit ellipse. This sum vector is, up to the constant negative factor $-(a-e)/2e$ the classical Runge-Lenz vector.
With this we come close to a central fact in Feynman's *Lost Lecture.* He considered a set of points on a Kepler ellipse such that the angle between neighboring position vectors is constant and he concluded that the corresponding points on the velocity diagram are *equidistant on this circle.* This follows since the vector from the endpoint of the (rotated) velocity vector to the midpoint of the circle is parallel to the position vector (from the sun to the planet).

Mathematically, the *parabolic* and *hyperbolic* Kepler orbits allow similar derivations of the $-1/r$-potential, which we will give next. Historically this played no role since the non-repeating orbits could not be determined with enough precision at the time.

Derivation of the $-1/r$-potential from a parabolic Kepler orbit. Let in the picture (below) $|p|$ be the distance from the sun at $(1/4, 0)$ to a tangent of a parabolic Kepler orbit and let $|v|$ be the orbital speed at that moment. Conservation of angular momentum says $p \cdot |v| = const.$ Let φ be the angle between the segment marked p and the vertical axis; since the sun is at the focal point of the parabola we have $p \cdot \sin\varphi = 1/4$. This and the previous angular momentum equation say that, up to a choice of unit for velocity, we have:

Kepler speed: $|v| = \sin\varphi$,

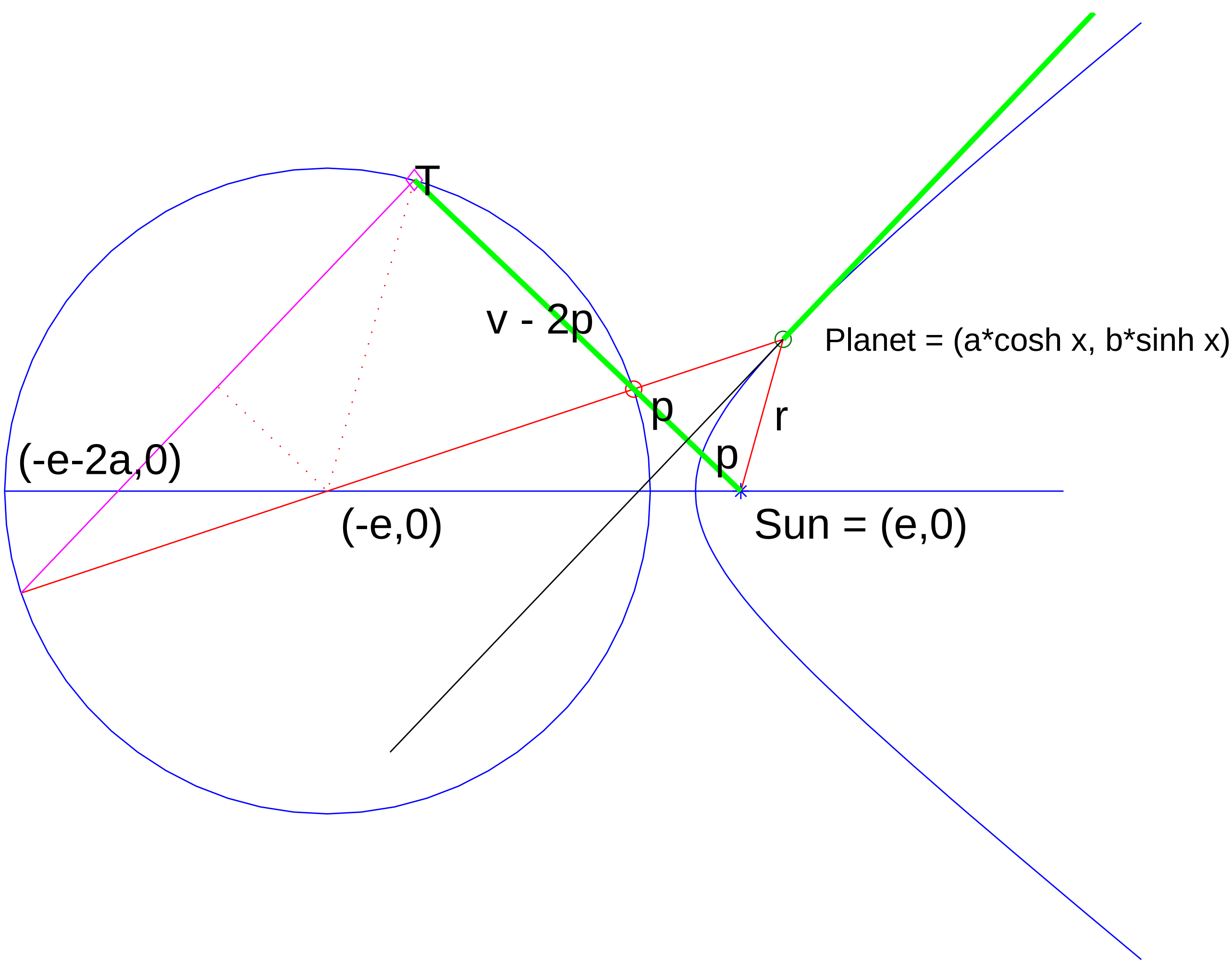
Angular momentum: $p \cdot |v| = 1/4$.



If we call r the distance to the planet, than we also have $p/r = \sin\varphi = |v|$. Multiplication with the angular momentum gives

$$\textbf{Kinetic energy: } \frac{1}{2}|v|^2 = \frac{1}{8r},$$
$$\textbf{Potential energy: } \frac{-1}{8r}.$$

Derivation of the $-1/r$-potential from a hyperbolic Kepler orbit.



As before we call p the distance from the sun to a tangent of the hyperbolic orbit and v the speed at that orbit point. Conservation of angular momentum says $p \cdot |v| = const.$ We use the property of the circle (radius $2a$) about products of segments on secants (which intersect at the sun S):

$$2p \cdot |T - S| = (2e - 2a)(2e + 2a) = 4b^2.$$

Therefore, again up to the unit for velocity, we have identified the correct

$$\textbf{Kepler velocity: } v = |S - T|.$$

Finally, similar triangles give:

$$p/r = (v - 2p)/4a \quad \text{or} \quad 4a/r = v/p - 2,$$

and elimination of p with the angular momentum, i.e. with $1/p = v/2b^2$, shows that kinetic energy plus a radial function are constant – thus identifying the $1/r$-potential:

$$4ab^2/r = v^2/2 - 2b^2.$$

Additional properties: As in the case of elliptical orbits we see that the *hodograph* is a circle because the velocity vector, rotated by 90 degrees, ends on the circle which we used for the construction of the hyperbola. And if we add to the endpoint of this rotated velocity a vector parallel to the position vector and of constant length 2a then we reach the midpoint of the circle, the other focal point of the orbit. The constant(!) difference vector between the two focal points, the geometric *Runge-Lenz vector*, differs from the common definition by the constant factor $(e - a)/2e$.