A Minimally Immersed Projective Plane*

See About Minimal Surfaces first.

This minimal surface is remarkable because its associate family contains an immersion of the triply punctured projective plane. It was discovered independently by Rob Kusner and Robert Bryant. It is also an example where the Weierstraß data can be explicitly integrated to a rational parametrization of the surface:

$$W_1(z) = \frac{i(z^2 - z^{-2})}{z^3 - z^{-3} + \sqrt{5}} = \frac{i(z^5 - z)}{z^6 - 1 + \sqrt{5}z^3}$$
$$W_2(z) = \frac{z^2 + z^{-2}}{z^3 - z^{-3} + \sqrt{5}} = \frac{z^5 + z}{z^6 - 1 + \sqrt{5}z^3}$$
$$W_3(z) = \frac{\frac{2i}{3}(z^3 + z^{-3})}{z^3 - z^{-3} + \sqrt{5}} - \frac{2i}{3} = \frac{2i}{3}\left(\frac{z^6 + 1}{z^6 - 1 + \sqrt{5}z^3} - 1\right)$$

The associate family is $F_\varphi(z) := \mathrm{Re}\,(e^{i\varphi}W(z))$.
Note the antipodal symmetry $W(-1/\bar{z}) = \overline{W}(z)$, which says that $F_0 = \mathrm{Re}(W) = -F_\pi$ is an immersion of the punctured projective plane. The poles of W are at the six zeros of the denominator, with $a^3 = \frac{3-\sqrt{5}}{2}$, $a \approx 0.72556$ they are $(a, -1/a) \cdot (1, e^{2\pi i/3}, e^{-2\pi i/3})$, three antipodal pairs.
The above formulas indeed represent minimal surfaces since

$$W_1'(z) + W_2'(z) + W_3'(z) = 0.$$

This is expressed as: $z \mapsto W(z)$ is a holomorpic null curve.

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

We also give the Weierstrass data:

$$g(z) = \frac{z^2(z^3 - \sqrt{5})}{\sqrt{5}\,z^3 + 1}, \text{ with } g(-1/z) = -1/g(z),$$

$$dh_z = \frac{2iz^2(z^3 - \sqrt{5})(\sqrt{5}\,z^3 + 1)}{(z^6 - 1 + \sqrt{5}\,z^3)^2} dz$$

For parameter lines we use conformal polar coordinates $z = \exp(u + iv)$. A band $-0.2 < u < 0$, $0 \le v \le 2\pi$ is mapped to a minimal Möbius band, see the Action Menu.

Near the punctures the minimal surfaces F_φ are asymptotic to planes. One says: these minimal surfaces have planar ends. At such planar ends $g \cdot dh$ or $1/g \cdot dh$ must have a double pole and g must have a branch point, i.e. $g'(z) = 0$ for finite z. Indeed, one can check that $g'(z_0) = 0$ at the zeros z_0 of the denominator of dh.

Planar ends have the following nice property: Invert the minimal surface in some sphere. Its planar ends invert to surface pieces with one-point holes. *These holes can be smoothly closed.* This gives immersions of the complete projective plane by inverting the minimal surface F_0 and closing the holes of its three inverted ends. The point at infinity inverts to a triple point of such immersions. These immersions are called Boy surfaces since a topological version was constructed in Boy's PhD thesis under Hilbert. It was the first immersion of the projective plane into $\mathbb{R}^3$. A model stands in front of the Oberwolfach Institute.

H.K.