About The Feigenbaum Tree*

See also: Julia Set of $z \to (z^2 - c)$

The Feigenbaum Tree is one of the earliest examples of parameter dependent behavior of a dynamical system. The dynamical system in question is called the *Logistic Map*:

$$f_\mu(y) := 4\mu \cdot y(1-y),\ y \in [0,1],\ \mu \in [1/4, 1].$$

Since both the parameter space, $[1/4, 1]$, and the dynamical space, $[0,1]$, are 1-dimensional, one can illustrate in a (μ, y)-plane how the dynamical behavior changes as the parameter μ varies. The usual experiment (and the one used in 3DXM) goes as follows: Starting with a set of initial values $\{y_k;\ y_k \in [0,1],\ k = 1, \ldots, K\}$ (and with as many parameter values μ as one wants to handle) one computes many iterations $f_\mu^{\circ n}(y_k), n = 1, \ldots, N$ with N large.

If one plots only the iterations with say $n \geq 1000$, then one observes in the (μ, y)-plane the *Feigenbaum Tree*: for small μ the iterated points $f_\mu^{\circ n}(y_k)$ converge to a stable fixed point of the map f_μ, $y_f = f_\mu(y_f)$, $y_f := 1 - 1/4\mu$. Observe that the derivative f' at the fixed point is $2 - 4\mu \leq 0$. At $\mu = 3/4$ the derivative at the fixed point is -1, so that the fixed point stops being attractive. It turns out that for larger μ the orbit of period 2 is attractive for a while

* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

– until μ reaches another bifurcation point after which an orbit of period 4 becomes attractive.

This period doubling "cascade" continues up to a certain μ-value, past which there is for a while no longer an attractive orbit. All this is clearly visible in the 3DXM demo. One should use the Action Menu entry: `Iterate Mouse Point Forward` to watch how arbitrary initial points are iterated and how these iterations converge to the attracting orbits of period 2^d in the left, period doubling, part of the Feigenbaum Tree. —*Speed-Up Note:* If one presses DELETE either during the default iterations or during the iteration of a point chosen by mouse, then all delays are skipped and the result of the iteration is reached quickly.

After the period doubling in the left part has been observed one wants to look at the right part of the Feigenbaum Tree more closely. The μ-interval which the illustration uses is the interval $[bb, cc]$. It can be changed in the Parameter entry of the Settings Menu. Since the attractive orbit of period 2 appears after $\mu = 0.75$, one loses only the simple attractors if one increases bb from 0.25 to 0.75, and one gains that the remaining part of the Tree is stretched by a factor of 3. In the same way one can magnify any part of the parameter space. Of course the dynamical space is always fully shown—unless one decides to use SHIFT+MOUSE to scale the image to see part of the dynamical space magnified. In this case translation using CONTROL+MOUSE-DRAG may be useful.

The most obvious feature in the right part of the Feigenbaum Tree are gaps, three fairly large ones and any number of thinner ones. The three large ones belong to parameter intervals where the map f_μ has attractive orbits of period 6, period 5, resp. period 3. If one magnifies a gap enough, one can experimentally check that the gaps belong to attractive orbits (use in the Action Menu `Iterate Mouse Point Forward`). One also observes that at the right end of these intervals the periods double again, and again. In other words, the Feigenbaum Tree illuminates, almost at the first glimpse, many properties of this 1-parameter family of iterated maps.

The Action Menu has been expanded by four entries `Iteration Invariant Density` (either with mouse choice of $aa = \mu$ or previous aa) and `Density Function` (again with mouse choice of aa or previous value). Before one chooses any of these one should look at `Iterate Mouse Point Forward`, where one sees how the iterated point, given by the vertical coordinate y, jumps around with fixed μ. The `Iteration Invariant Density` expands this: 1000 different y-values are chosen and represented in the left-most column on the screen. These points are iterated and shown in the second column, iterated again and shown in the third column, and so on, 400 times. Except for the first few columns one clearly sees a density pattern develop: all the vertical columns look essentially alike. This can be studied further with the entry `Density Function`: Here we count how often each pixel-sized interval of the dynamical (=vertical) interval is visited during the iterations and we plot the counting result (normalized to fit on the screen). We observe a function that describes the probability density with which each pixel interval is visited. – These demos explain why the curves that represent attractors do extend into the chaotic regions.

Finally we remark that the Feigenbaum Tree is related to the real part of the Mandelbrot set because the Mandelbrot set also parametrizes quadratic maps $z \to f_c(z) := (z^2 - c)$ according to their dynamical properties. If c is chosen from the big bottom apple then f_c has an attractive fixed point. As one passes on the real axis from the apple to the disk above it, the fixed point changes from attractive through indifferent to unstable and the orbit of period 2 becomes attractive. As one moves (always along the real axis) towards the top of the Mandelbrot set one continues to meet exactly the same kind of dynamical behavior as one sees in the Feigenbaum Tree. For more details see the documentation for *Julia Set of* $z \to (z^2 - c)$.

H.K.