Surfaces of Revolution*
with Constant Gauß Curvature

A surface of revolution is usually described by giving its meridian curve $s \mapsto (r(s), h(s))$. The surface is then obtained by rotation:

$$(x, y, z)(s, \varphi) := (r(s)\cos\varphi,\ r(s)\sin\varphi,\ h(s)).$$

Any kind of curvature condition can be expressed as an ordinary differential equation for the meridian curve.

The case of given Gauß curvature $K(s)$ is particularly simple if the meridian is parametrized by arclength, i.e., $r'^2 + h'^2 = 1$. In this case the meridian is, under the condition $|r'(s)| < 1$, determined by

$$r''(s) + K(s) \cdot r(s) = 0, \qquad h(s) = \int_0^s \sqrt{1 - r'(t)^2}dt.$$

In the case of $K(s) := 0$ each solution $r(s)$ is a linear function. We obtain circular cylinders and cones.
Ferdinand Minding (1806 - 1885) studied the case of constant nonzero curvature. He determined the geodesics on the Pseudosphere ($K = -1$), but we could not find whether he was the first to describe this famous surface. He obtained the following formulas:

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

The examples for $K = 1$:

The Sphere:

$$r(s) = \sin(s), \quad 0 \le s \le \pi$$

With cone points:

$$r(s) = a\sin(s), \ 0 \le s \le \pi, \ 0 < a < 1$$

With singularity curve:

$$r(s) = a\cos(s), \ -b \le s \le b, \ a > 1, \ \sin(b) = 1/a.$$

The examples for $K = -1$:

The Pseudosphere (with singular curve $\{s = 0\}$):

$$r(s) = \exp(-s), \ 0 \le s$$

With cone point and singular curve:

$$r(s) = a\sinh(s), \ 0 \le s \le b, \ 0 < a < 1, \cosh(b) = 1/a$$

With two singular curves:

$$r(s) = a\cosh(s), \ -b \le s \le b, \ 0 < a, \ \sinh(b) = 1/a.$$

The Pseudosphere cannot be extended beyond the singular curve $\{s = 0\}$, but the metric of the Pseudosphere,

$$ds^2 + \exp(-2s)d\varphi^2,$$

extends to a complete metric on $\mathbb{R} \times \mathbb{R}$ with $K = -1$ and the same is true for the metric of the last example:

$$ds^2 + a^2\cosh^2(s)d\varphi^2.$$

The fact that such a simply connected complete Riemannian plane with $K = -1$ is isometric to the non-Euclidean

geometry of Bolyai and Lobachevsky was not yet established when Minding studied geodesic triangles on the Pseudosphere.
Surfaces with $K = -1$ in $\mathbb{R}^3$ have another interesting property: If the asymptote directions are described by two vectorfields of *constant* length, then these vector fields commute. For parametrizations of $K = -1$ surfaces with asymptote lines as parameter lines one therefore has: the edge lengths of the parameter quadrilaterals are all the same. Such nets are called Tchebycheff nets. They are determined by the angle between the asymptote lines. A quite unexpected theory developed from here. It is outlined in
`About Spherical Surfaces` (see the Documentation Menu).
The Pseudosphere turns out to be the simplest example of that theory.
These Tebycheff nets also play a crucial role in the proof of Hilbert's theorem, stating that the hyperbolic plane cannot be smoothly immersed isometrically into $\mathbb{R}^3$.

H.K.