About ODE 2nd Order: Charged Particles

The Motion of Charged Particles in Magnetic Fields

The path $p(t)$ of a particle with electric charge e and mass m in a magnetic field B is given by

$$m \cdot p''(t) = e \cdot p'(t) \times B(p).$$

(The right hand side is called the *Lorentz Force*.)

This implies that, for an arbitrary magnetic fields, B, the kinetic energy $E(t) = \frac{m}{2}\langle p',\ p'\rangle(t)$ is constant in time.

One should first convince oneself in the case of a

Constant Magnetic Field

that a particle can move tangentially to the field lines, in circles around the field lines and in helices around the field lines, i.e., in any linear combination of the first two special cases.

Put in *Settings, ODE Settings*:

$v_x = 0.003, v_y = 0.003, v_z = 0.5$, to obtain almost circles around the field lines.

And put $v_x = 0.2, v_y = 0.2, v_z = 0.001$ to obtain approximate straight lines parallel to the field lines. The *Default Settings* give a general helix. We will also try to understand charged particle motions in nonlinear fields by looking at such special cases.

We consider next motion of a charged particle in the:

Field of an Electric Current

along the x-axis. The field lines are circles parallel to the y-z-plane with centers on the x-axis. In this case, the *Default Settings* give initial conditions in the x-y-plane (a symmetry plane of the field) that are orthogonal to the field lines. The solution curves therefore remain in this plane, and are, for small velocities, *almost circles around the field lines.* But, because the absolute value of the field is decreasing with r, these solution curves are more strongly curved the nearer they are to the wire. They are therefore rolling curves with a translational period in the direction of the wire. (See *Plane Curves, Cycloid* and put $aa = 1$ and $bb = 6.5$ in the *Set Parameters* dialog.) If in *ODE Settings* one increases the velocity to $v_y = 0.5$, then the translational part is so large that the consecutive loops do not intersect.

We obtain solution curves which *almost follow the field lines* if we make the initial velocity tangential to the field lines and fairly small:

$v_x = 0, v_y = 0, v_z = 0.02,$
Time span $= 450$, Step-size $= 0.2$.

Now slowly increase v_z, e.g., to $v_z = 0.2$, to obtain another family of solutions *follows the field lines, but winding around them in small loops.*
Next in *Settings, ODE Settings*, put:
$v_x = 0.02, v_y = 0.02, v_z = 0.01$
leaving Time span $= 450$, Step-size $= 0.2$, as before.
Finally we increase the initial velocity to obtain solution curves that look fairly wild at first but can be seen to follow the pattern which we recognized for more special initial conditions, namely put
$v_x = 0.2, v_y = 0.2, v_z = 0.1$
to see solutions that follow helices with wide loops around them. *Try by all means to view this in stereo!*

Finally we consider the so-called *Störmer Problem*, namely the motion of charged particles in a Magnetic Dipole Field. Since the magnetic field of our Earth is a dipole field, such motions occur in the van Allan Belt when charged particles from the Sun's plasma meet the Earth. A dipole field $B(p)$ with a magnetic moment mm is given by:

$$B(p) = 3\langle mm,\ p\rangle \frac{p}{|p|^5} - \frac{mm}{|p|^3}.$$

The *Default ODE Settings* give a fairly general but somewhat complicated solution curve. To see solutions that *almost follow the field lines* use ODE Settings to set a small initial velocity tangential to the field lines, say $v_x = 0, v_y = 0, v_z = 0.05$. To see solutions that *almost circle the field lines in the equator plane of the dipole*, in the ODE Settings dialog, choose small initial conditions in the equator plane, e.g., $v_x = 0.1, v_y = 0.1, v_z = 0$. The resulting curves are close to rolling curves. (Compare *Plane Curves, Circle* using, Parameter Settings: $hh = -0.125$, $ii = 4$, and increase t-Resolution to 200, then choose Generalized Cycloids from the Action Menu.) Since the absolute value of the field increases along the field lines from the equator towards the poles, one cannot have solutions that almost follow the field lines while circling them in narrow loops, however one can approximate such behavior with the initial condition $v_x = 0.035, v_y = 0.035, v_z = 0.05$.

In a Plenary address on Dynamical Systems he gave at the 1998 International Congress of Mathematicains in Berlin, Jürgen Moser had an interesting discussion of the Störmer Problem that we reproduce below from Documenta Mathematica, Extra Volume, ICM 1998, pp. 381–402. (After the lecture, one of us approached Moser and showed him the visualization of the Störmer Problem in 3D-XplorMath. He appeared to be delighted by it, but said something looked not quite right to him, a remark that helped us eliminate a small bug!)

Here is the extract from Moser's lecture.

R.S.P. & H.K.

d) The Störmer Problem.

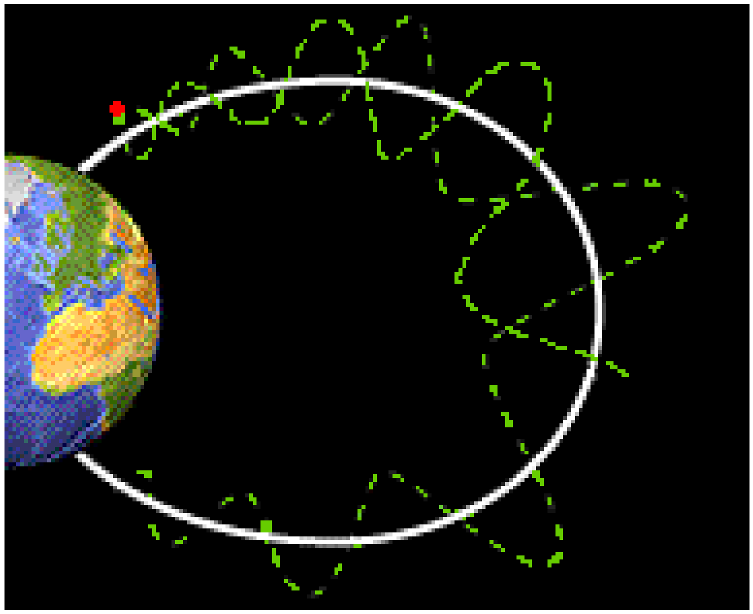
Another large scale confinement region is known in the magnetic field of the earth. With the advent in 1957 of satellites it was soon discovered that the earth was surrounded by (two) belts of charged particles caused by its magnetic field. Since the beginning of the century it was known that such charged particles were present above the atmosphere and were responsible for the aurora borealis (and australius). It was Störmer (incidentally president of the ICM 1936 in

Oslo) who made calculations of the orbits of these charged particles moving in the magnetic field of the earth, which he modelled as a magnetic dipole field. This is an interesting nonlinear Hamiltonian system.

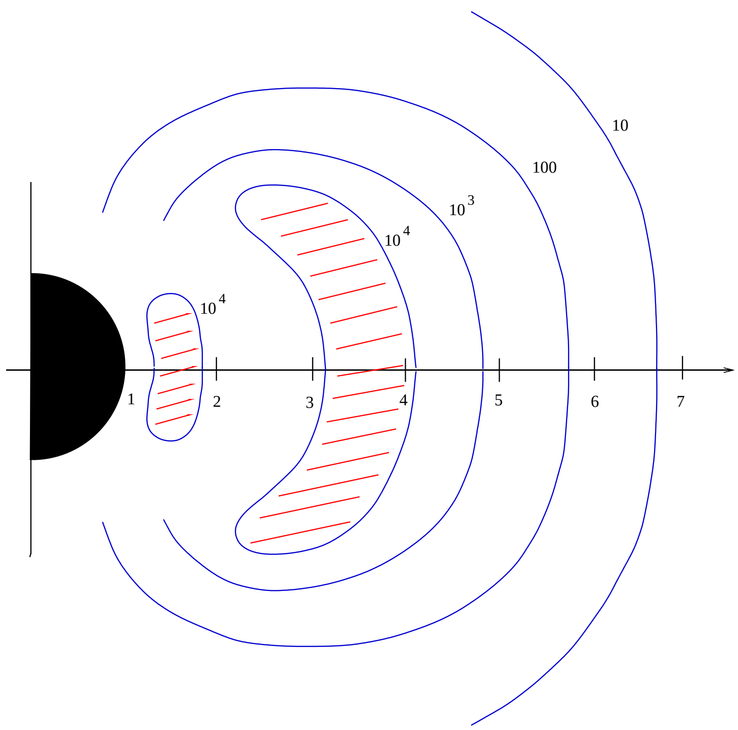
The satellite measurements led to the discovery of two regions surrounding the earth, the so-called van Allan belts, in which charged particles were trapped. It turns out that it is an example of a magnetic bottle to which the stability theory is applicable (M. Braun 1970).

It is interesting to realize the dimensions involved: For electrons, the "cyclotron radius" is of the order of a few kilometers and the corresponding periods of oscillation about one millionth of a second! The "bounce period" of travel from north pole to south pole and back is a fraction of a second.

In addition to these natural van Allan belts several artificial radiation belts have been made by the explosion of high-altitude nuclear bombs since 1958. Some of those so created belts had a lifetime up to several years—which shows the long stability of these experiments as well as the irresponsibility for carrying them out! Some 30 years ago these tests have been stopped.



Störmer problem



Van Allan belt