ODEs with Central Forces*

This category models the motion of a particle moving according to Newton's Law "$F = ma$", where the force F is a "central" force, i.e., one directed towards (or away from) the origin and whose magnitude depends only on the distance of the point from the origin. (As we will recall below, this also models the motion of two particles in space when the force between them is directed along the line joining them and has a magnitude that depends only on their separation.)

By "conservation of angular momentum", the particle moves in a fixed plane, and this is taken as the plane of the screen, with the origin at the screen center. By default, the x and y axes are visible but, as usual, you can reverse this using the View Menu.

There are several pre-programmed central forces:

Coulomb: $F(r) = -aa/r^2$ (default: $aa = 2.5$)

Power Law: $F(r) = -aa\, r^{bb}$ (defaults: $aa = 2.5$, $bb = -2.01$)

Yukawa: $F(r) = -aa \exp(-bb\, r)(bb/r + 1/r^2)$ (defaults: $aa = 2.5, bb = 0.25$)

Hooke's Law : $F(r) = -aa\, r$ (default: $aa = 2.5$)

Higgs: $F(r) = aa\, r(1 - (r/cc))$ ($aa = 3.0, cc = 2$).

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

(And there is also a provision for a "User Defined" central force law.)

After choosing one of the pre-programmed examples, an orbit will be displayed for a default initial position *(xinitial,yinitial)* and initial velocity *(xdotinitial,ydotinitial)*. The time evolution of the orbit will continue until the mouse is clicked. The same initial position $(0.5, 0.5)$ and same initial velocity $(-1.25, 2.0)$ is used for the Coulomb's law, the Power Law, and Yukawa Law. (This, in particular, demonstrates the fact that having all bounded orbits closed is rather special to the case of a power law with exponent -2.)

To display an orbit with different initial conditions, choose "IC by mouse (Drag)" from the Central Force menu and then click to choose an initial value, and drag to choose an initial velocity—the velocity vector will be proportional to the vector difference of the MouseUp and MouseDown points. (If instead you choose "IC by mouse (Throw)" from the menu, then the initial point is again the MouseDown point, but the velocity is computed using the position of the mouse one-half second after MouseDown rather than the MouseUp position. (In either case, the new orbit will start being displayed immediately, and continue to evolve until the mouse is again clicked). There is still a third way to choose initial conditions; namely, choose "ODE Settings..." from the Settings menu. This will display a Dialog in which you can choose the initial values of x and y. In addition x- and y- components of velocity.

(In this Dialog, you can also set the value of the mass, the step-size used in the Runge-Kutta integration, and the "dotted-ness" of the orbit. You can also change the values of any parameters on which the chosen central force depends by choosing Set Parameters... from the Settings menu. (To check dependence of the current central force on the parameters $aa, \ldots, ii$, choose either "About this Object" from the Central Force menu.) After making changes to the parameters, the mass, initial conditions, stepsize, and dotted-ness of the orbit (see below for more about the latter two) choose Create from the Central Force menu to start the display of the new orbit. (If you want the orbit drawn onto a clean screen, choose Erase before choosing Create.)

Instead of using one of the pre-programmed central forces, the user can define a central force law by choosing User Defined... in the Central Force menu and entering a function of the radius, r, and the usual usual nine parameters $aa, \ldots, ii$. (After clicking the OK button, either choose an initial position and velocity with the mouse as described above, or use the Set Parameters... and ODE Settings... and then choose Create.)

If the Time step-size is set equal to zero in the ODE Settings Dialog, then the stepsize will be set by an adaptive algorithm that makes the stepsize (roughly) inversely proportional to the magnitude of the velocity. This has the advantage of making the stepsize very small where the velocity is large, avoiding numerical errors that are otherwise

inherent in such situations. The downside is that all timing information is lost—the orbit is traced out equal lengths in equal times rather than by Kepler's Law of equal areas in equal times.

The Dot Spacing in the ODE Settings... Dialog sets the time between which successive dots are drawn along the orbit. If Dot Spacing is set equal to zero, then the orbit is drawn as a solid curve.

Finally, we recall that the real significance of a central force is that it models the Newtonian motion of a two-particle system (say with masses m_1 and m_2) in which the force $F(r)$ on each particle is equal and opposite to that on the other, is along the line joining the two particles and has a magnitude depending only on their distance of separation, r. First, by conservation of linear momentum, the center of mass of the two particles moves with constant velocity in a straight line, so making a Gallilean coordinate transformation we can assume the center of mass is fixed at the origin. The problem then is to find the time evolution of the vector $\vec{r}$ joining particle one to particle two, and it is easily seen that $\vec{r}$ satisfies Newton's equations for a particle moving in response to the central force $F(r)$ and with mass the harmonic mean of m_1 and m_2 (the so-called "reduced mass"). For details see any good text on classical mechanics, for example, Chapter 3 of Goldstein.

R.S.P.