Minimal Surfaces: Catalan, Henneberg, Scherk*

These surfaces are early discovered minimal surfaces. They were found as explicitly parametrized surfaces, while soon afterwards variations of the Weierstrass representation became the main tool of description. This changed again in the early 1930s when Douglas and Rado solved the Plateau Problem with Functional Analysis methods. The Weierstrass representation had, after the work of Ossermann, a comeback in the 1980s.
Scherk's doubly periodic minimal surface (1835):

$$x(u,v) := \frac{u}{bb},\ y(u,v) := \frac{v}{bb}\ z(u,v) := \frac{1}{bb}\ln(\frac{\cos(v)}{\cos(u)}).$$

Catalan's minimal surface (1855), associate family:

$$\begin{aligned} x(u,v) &:= \frac{1}{bb}(\cos(aa\cdot\pi)(u-\sin(u)\cosh(v))+ \\ &\qquad \sin(aa\cdot\pi)(v-\cos(u)\sinh(v))-4) \\ y(u,v) &:= \frac{1}{bb}(\cos(aa\cdot\pi)(1-\cos(u)\cosh(v))+ \\ &\qquad \sin(aa\cdot\pi)\sin(u)\sinh(v)) \\ z(u,v) &:= \frac{1}{bb}(\cos(aa\cdot\pi)4\sin(u/2)\sinh(v/2)+ \\ &\qquad \sin(aa\cdot\pi)4\cos(u/2)\cosh(v/2)). \end{aligned}$$

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

Henneberg's minimal surface (1875), associate family:

$$caa := \frac{2\cos(aa\cdot\pi)}{0.01+|bb|},\ \ saa := \frac{2\sin(aa\cdot\pi)}{0.01+|bb|}$$

$$\begin{aligned}
x(u,v) :=& caa\cdot(\sinh(u)\cos(v)-\frac{1}{3}\sinh(3u)\cos(3v))\\
&+saa\cdot(\cosh(u)\sin(v)-\frac{1}{3}\cosh(3u)\sin(3v))\\
y(u,v) :=& caa\cdot(\sinh(u)\sin(v)+\frac{1}{3}\sinh(3u)\sin(3v))\\
&+saa\cdot(-\cosh(u)\cos(v)-\frac{1}{3}\cosh(3u)\cos(3v)+\frac{4cc}{3})\\
z(u,v) :=& caa\cdot(\cosh(2u)\cos(2v)-cc)\\
&+saa\cdot\sinh(2u)\sin(2v)\\
&u\in\mathbb{R}, v\in[-\pi,\pi],\ \text{a cylinder domain.}
\end{aligned}$$

The default value of the translation parameter, $cc = 0$, puts the symmetry point of Henneberg's surface at the origin. If one wants to scale up a neighborhood of the branch point at $u = 0$, $v = 0$, use $cc = 1$ to put that branch point at the origin and make use of the scaling parameter bb in the denominator.

Scherk's discovery of the above doubly periodic surface, of its singly periodic conjugate surface and of three less spectacular ones was a sensation since the only other known minimal surfaces, the catenoid and the helicoid, were already 50 years old. Scherk's surfaces were destined to play a major role in the discovery period after 1980.

Catalan's surface was next, 20 years later. This slow progress reflects the fact that no methods for the construction of minimal surfaces were known. This changed with Riemann, Weierstrass, Enneper and finally Schwarz. Their methods built on complex analysis and allowed to write down arbitrarily many examples. Therefore the emphasis shifted to examples that had additional properties. In particular complete, embedded minimal surfaces were sought. Quite a few triply periodic embedded ones were found, but it took another 100 years before the next embedded finite total curvature example after the catenoid was constructed: Costa's example, a minimal embedding of the thrice punctured square torus.
Henneberg's minimal surface was studied a lot because of its two branch point singularities (on the z-axis). Such singularities are difficult to imagine and Henneberg's surface is a simple example to exhibit them. The default image in 3DXM shows a small neighborhood of the two branch points. The segment between them is a selfintersection line of the surface. The lines $\{x = \pm y, z = 0\}$ lie also on the surface. $180°$ rotation around any of these straight lines is a symmetry of the surface and the conjugate surface has, correspondingly, the three planes which pass through the origin and are orthogonal to one of these three lines, as symmetry planes. - Note: The conjugate surface has twice the area of Henneberg's surface. This is because Henneberg's surface is covered twice. For about 100 years it was the only known non-orientable minimal surface.

The last entry in the Action Menu emphasizes a `Möbius band` on Henneberg's surface. The default morph is the associate family morph.
The `Range Morph` (from the Animate Menu) shows larger and larger pieces of the surface - scaled down to fit on the screen. One can see still larger portions by increasing `uMax` beyond 0.95. (Choose $b1 > 1$ to compensate for the growing size.)
A `third morph` expands a band around the two branch points and moves it over the surface.
The parameter line $u = 0, \ v \in [-\pi, \pi]$ is a symmetry line of the domain cylinder. It is mapped to the segment between the branch points on the z-axis and 180° rotation around it is a symmetry of the surface. The two lines $v = 0$ and $v = \pi$ are also symmetry lines of the domain cylinder. They are mapped to curves of reflectional symmetry, reflection in the x-z-plane. They end in the branch points like a Neil parabola, $\{x^3 = z^2\}$.

H.K.