Complex Map $z \to \log z$
The Complex Logarithm

Look at the function $z \to e^z$ and its ATO first.

The complex Logarithm tries to be the inverse function of the complex Exponential. However, exp is $2\pi i$-periodic, so such an inverse can only exist as a multivalued function.

From the differential equation $\exp' = \exp$ follows that the derivative of the inverse is not multivalued and in fact very simple:

$$\log'(z) = 1/z.$$

Integration of the geometric series

$$\begin{aligned} 1/z &= 1/(1-(1-z)) = \textstyle\sum_k (1-z)^k \\ &= \big(\textstyle\sum_k -(1-z)^{k+1}/(k+1)\big)' \end{aligned}$$

gives the Taylor expansion around 1 of log. The so called "principal value" of the complex Logarithm is defined in the whole plane, but slit along the negative real axis, for example by integrating the derivative $\log'(z) = 1/z$ in that simply connected domain along any path which starts at 1.

Different values of $\log z$ differ by integer multiples of $2\pi i$, e.g. $i = \exp(\pi i/2)$ implies $\log i = \pi i/2 + 2\pi i \cdot \mathbb{Z}$.

H.K.