Projected Sphere*

The elementary and differential geometric properties of triangles and curves on spheres are explained in “About Spherical Curves” in the documentation for Space Curves.

This exhibit has two goals:
1) The mechanical constructions of *ellipses, cycloids* and *lemniscates* in the plane are repeated on the sphere to give mechanical constructions of *spherical ellipses, spherical cycloids* and *spherical lemniscates.* In the plane and on the sphere these constructions also give the tangents of the curves. The curves are chosen in the Action Menu of the sphere exhibit. They have the same parameters as in the space curve exhibits.

2) Two famous maps are explained: The angle preserving stereographic projection from the sphere to a tangent plane and Archimedes’ area preserving projection from a circumscribed cylinder to the sphere.
The sphere can be rendered with four different grids, selectable in the Action Menu. After standard (geographical) polar coordinates have been chosen, one can add (also from the Action Menu) four different spherical curves or Archimedes’ projection. The spherical ellipses can also be viewed on a grid consisting of confocal ellipses and standard longitudes. Stereographic projection is best explained on a grid whose parameter lines are two orthogonal families of circles through the projection center (”South Pole”);

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

these circles are mapped to lines parallel to the x-axis or the y-axis. The parameter lines of the last grid are two families of "loxodromes", curves which meet the longitudes under constant angles; with this grid one can see – via stereographic projection – that groups of Möbius transformations may look much simpler on the Riemann sphere than in the Gaussian plane.

Which parameters control what?

Spherical Ellipses: bb = distance of focal points, $aa > bb/2$ = major axis. The default morph changes aa. For the elliptic grid also: bb = distance of the focal points. The construction demo runs automatically; if $0 < ff < 1$ is taken as morphing parameter, the demo runs as movie.

Spherical Cycloids: $aa \cdot \pi$ = radius of the fixed circle. Integer ee = order of symmetry (determines the radius of the rolling circle). Length of drawing stick = $bb \cdot$ radius of rolling circle. Automatic demo, morphing parameter ff.

Spherical Lemniscates: cc = length of the middle rod, dd = length of the two outer rods. ff parameter of the drawing pen, $ff = 0.5$ midpoint of middle rod. Automatic demo, can be saved with morphing parameter $0 < ee < 1$.

Loxodromes, Loxodromic grids: aa = slope parameter, with $aa = 0$ giving a longitude curve resp. the standard polar grid. cc = parameter (in degrees) for rotating the sphere around the x-axis - if also stereographic projection is chosen, then its center and image plane stay fixed while the sphere rotates.

Explanation of Archimedes' Projection

Let $-\pi/2 \leq u \leq \pi/2$ be the latitude parameter on a unit sphere parametrized as $(\sin(u)\cos(v),\ \sin(u)\sin(v),\ \cos(u))$. Add the tangential cylinder around the equator ($u = 0$), parametrized as $(\cos(v),\ \sin(v),\ \cos(u))$. Archimedes' map is horizontal radial projection, i.e. points with the same parameters (u, v) on the cylinder and the sphere correspond to each other. This map shortens latitudes of the cylinder at height $\cos(u)$ by the factor $\sin(u)$. The vertical lines on the cylinder are, locally at height $\cos(u)$, stretched by the factor $1/\sin(u)$.
With formulas:

$$\begin{aligned} &\text{cylinder:} && \left|\frac{d}{du}(\cos(v), \sin(v), \cos(u))\right| = \sin(u), \\ &\text{sphere:} && \left|\frac{d}{du}(\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))\right| = 1. \end{aligned}$$

This shows that Archimedes' map is area preserving. The unit sphere has therefore the same surface area as the cylinder of height 2 and radius 1, namely $2 \cdot 2\pi$.

This also implies that the volume of the solid sphere is $4\pi/3$ because we can partition the boundary sphere into very small domains and view the solid sphere as the (disjoint) union of height$= 1$ pyramids which have as base the small domains and their vertices are at the center of the sphere.

Explanation of Stereographic Projection

Geometrically stereographic projection is defined as the 1-1 central projection from a point on the sphere to the tangent plane at the opposite point and vice versa. The same definition works for the 2-sheeted hyperboloid, which gives a map of similar importance as in the spherical case. Formulas are slightly simpler if one projects to the plane through the center of the sphere which is parallel to the tangent plane opposite the center. Take $S = (0, 0, -1)$ as projection center, then:
Let $(x, y, z) \in \mathbb{S}^2$, i.e. $x^2 + y^2 + z^2 = 1$. For $z \neq -1$ put

$$St(x, y, z) := (x, y, 0)/(1 + z).$$

The inverse map is used just as often:
Let $(u, v, 0) \in \mathbb{R}^2 \subset \mathbb{R}^3$, then with $r^2 := 1 + u^2 + v^2$ put

$$St^{-1}(u, v, 0) := (2u,\ 2v,\ 2 - r^2)/r^2.$$

The most important property of stereographic projection is:

This map preserves angles or shorter: *it is conformal.*

Recall: A map preserves angles if for any two intersecting curves in the domain holds: *Their intersection angle is the same as the intersection angle of their image curves.* Of course, the intersection angle of curves is defined as the intersection angle of their tangents at the intersection point. To prove that angles are preserved, it is therefore enough to take as 'curves' in the image plane just all straight lines. Their stereographic preimages on the sphere are circles through the projection center S. Any two such circles in-

tersect at S and their other intersection point with the *same angle.* For any such circle holds: Its tangent at S and its image line in the opposite tangent plane are *parallel.* In other words: *The image lines of two circles through S intersect with the same angle as the two circles.*
This proves that stereographic projection is conformal.

A second basic property of stereographic projection is:
Circles on the sphere which do not pass through S are mapped to circles in the image plane. (We know already that circles through S are mapped to straight lines.)
Proof. Take a circle on the sphere and consider all tangents to the sphere which are perpendicular to the circle. They form the tangential cone which touches the sphere along the given circle. For each of these tangents consider the plane spanned by the considered tangent and the projection center S. Intersect these planes with the sphere to get circles through S which all intersect the given circle orthogonally and all pass through one point, namely the intersection of the line through S and the vertex of the tangential cone with the sphere. The stereographic images of these circles are all straight lines which all pass through one point and meet the image curve orthogonally, i.e. they are like radii. The image curve is therefore a circle – either by the uniqueness theorem for ODEs or because the cone with vertex S through the given circle is a quadratic cone so that the image curve is known to be an ellipse - which in turn has to be a circle because of the orthogonal radii.
One can also prove this fact by computation, of course.