The Area Preserving Henon Twist Map*

User Defined Example

The User Defined entry is designed to study the behaviour of 2-dimensional maps under forward iteration near an isolated, neutral fixed point. (We want a fixed point inside the window since otherwise most of the iterated points will move out of sight.) Our example is Henon's quadratic, area preserving twist map F:

$$F(x,y) := \begin{pmatrix} \cos(aa)\cdot x - \sin(aa)\cdot(y - e^{bb}\cdot x|x|) \\ \sin(aa)\cdot x + \cos(aa)\cdot(y - e^{bb}\cdot x|x|) \end{pmatrix}.$$

Henon used x^2 instead of $x|x|$ for the perturbation term. See below.

The main parameter aa controls the derivative of F at the fixed point $(0,0)$; $\mathrm{d}F|_{(0,0)}$ is the rotation matrix with angle aa. The behaviour of the iterations changes strongly with aa. Try also $-aa$. F is area preserving since the Jacobian determinant $\det(\mathrm{d}F) = 1$ everywhere.

By default $e^{bb} = 1$. This parameter serves to choose the size of the neighborhood of the fixed point, because of the scaling property

$$F(\vec{x}; e^{bb}) = e^{-bb}\cdot F(e^{bb}\cdot\vec{x}; 1).$$

We use $\exp(bb)$ instead of bb, because the scaling parameter is a multiplicative rather than an additive parameter.

---

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

The iteration is applied to the segment $[0,1]\cdot(cc,dd)$. The number of points on this segment is *tResolution*. The default number of iterations is $ee = 2000$. The next 2000 iterations are obtained from the Action Menu Entry: `Continue Curve Iteration`.
Since the graphic rendering is much slower than the the computation of iterations one can increase the parameter hh from its default value $hh = 1$ and then only one out of hh iterations is shown on the screen. This is useful if one needs to see the result of a large number of iterations. (For example $hh = 4 \cdot n$ in the case $aa = \pi/2$.)

The Action Menu Entry `Iterate Mouse Point Forward` allows to iterate a single point. During the selection the point coordinates appear on the screen. If DELETE is pressed during the iteration then the waiting time at each step is cancelled so that the point races through its orbit.

The Action Menu Entry `Choose Iteration Segment By Mouse` allows to Mouse-select initial and final point of a segment on which ff points will be distributed and iterated (by default $ff = 16$). The parameter hh speeds up the iteration as above. After the first ee iterations an Action Menu Entry is activated and allows to iterate further.

As ususal one can `translate` the image by dragging or one can `scale` it by depressing SHIFT and dragging vertically.

One can also `morph` the images. They change rather drastically with aa. As default morph bb is decreased so that the neighborhood of the fixed point gets expanded. One observes that most of the iterated points travel on *invariant curves* around the fixed point. Occasional periodic points clearly show up in the image. If aa is an irrational multiple of π then the visible periods do increase as the neighborhood of the fixed point expands with decreasing bb. (For the default morph the number ee of iterations is restricted to 500 to reduce waiting times.)

The Henon twist map can be written as a rotation plus a quadratic perturbation:

$$F(x,y) := \begin{pmatrix} \cos(aa) & -\sin(aa) \\ \sin(aa) & +\cos(aa) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \text{perturb},$$

$$\text{perturb} := e^{bb} \cdot x|x| \cdot \begin{pmatrix} +\sin(aa) \\ -\cos(aa) \end{pmatrix}.$$

The scalar product between the perturbation and the tangent to the rotation circles is the

$$\begin{aligned} \text{Forward Perturbation } &= \\ &-e^{bb}|x|^3 \cdot (\sin^2(aa) + \cos^2(aa)). \end{aligned}$$

This explains why we changed the Henon map. Our negative forward perturbation means that the images under F stay behind the rotation image, and more so the larger $|x|$. This is the usual behavior of a monotone twist map.

Henon's perturbation has the factor x^3 instead of $|x|^3$, so that the twist in the left half plane partially cancels the twist in the right half plane. In our definition do the elliptical islands around periodic points appear more easily, while with Henon's definition the behaviour near the fixed point, in the case when aa is a rational multiple of π (e.g. $aa = \pi/2$), is much more complicated.

We recommend that users try out also Henon's definition and definitions of their own.

H.K.