About Spherical Ellipses

See ATO for Planar Ellipses

In 3DXplorMath the *Default Morph* shows a family of ellipses with fixed focal points F_1, F_2 as the larger axis aa varies from its allowed minimum $e = bb/2$ to its allowed maximum $\pi - e = \pi - bb/2$. Another interesting morph is $0.11 \leq aa \leq 1.43, \quad 0.2 \leq bb \leq \pi - 0.2$: the distance of the focal points increases until they are almost antipodal and the major axis is only slightly longer than the distance of the focal points.

Elementary Definition. Many elementary constructions from planar Euclidean geometry have natural analogues on the twodimensional sphere $\mathbb{S}^2$. For example, we can take the definition of planar ellipses and use it on the sphere as follows: Pick two points $F_1, F_2 \in \mathbb{S}^2$ of spherical distance $2e := dist(F_1, F_2) < \pi$ and define the set of points $P \in \mathbb{S}^2$ *for which the sum of the distances to the two points* $F_1,\ F_2$ *equals a constant* $=: 2a$, i.e. the set:

$$\{P \in \mathbb{S}^2;\ dist(P, F_1) + dist(P, F_2) = 2a\},$$

to be a Spherical Ellipse.

In the Euclidean plane there is only one restriction between the parameters of an ellipse: $2e < 2a$. Since distances on $\mathbb{S}^2$ cannot be larger than π we have two restrictions in spherical geometry: $2e < 2a < 2\pi - 2e$.

For fixed focal points, i.e. for fixed e, these curves cover the sphere (we allow that the smallest and the largest ellipse degenerate to great circle segments). One observes that the ellipse with $2a = \pi$ is a great circle and that ellipses with $2a > \pi$ are congruent to ellipses with $2a < \pi$ and focal points $-F_1, -F_2$. This is because $dist(P, F) = \pi - dist(P, -F)$ implies

$$\pi < 2a = dist(P, F_1) + dist(P, F_2) \Rightarrow$$
$$dist(P, -F_1) + dist(P, -F_2) = 2\pi - 2a < \pi.$$

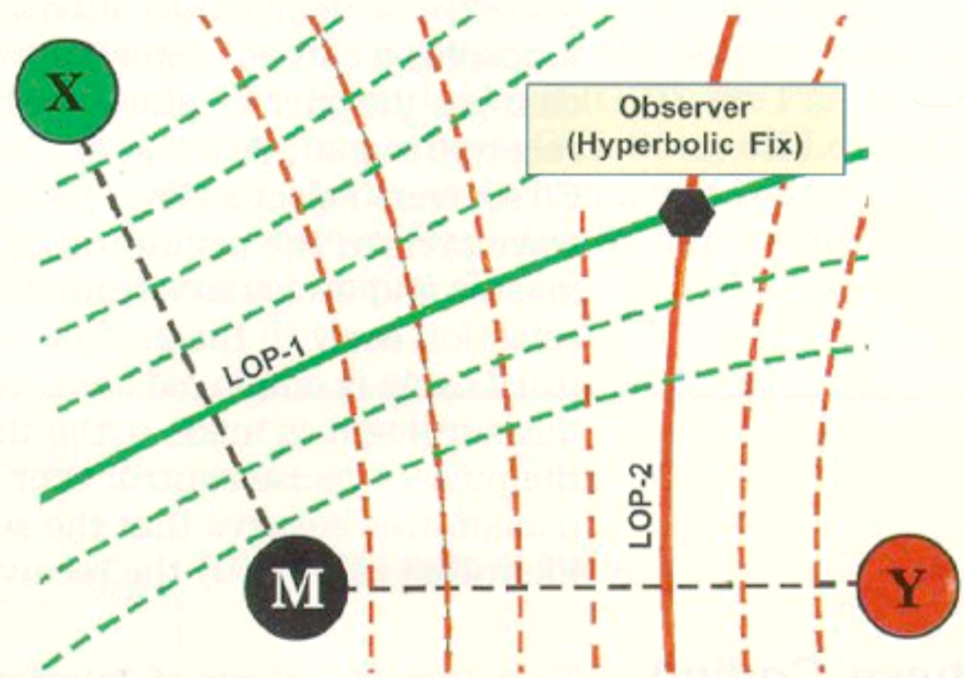
Similarly, on the sphere one does not need to distinguish between ellipses and hyperbolas:

$$\{P \in \mathbb{S}^2;\ dist(P, F_1) + dist(P, F_2) = 2a\} =$$
$$\{P \in \mathbb{S}^2;\ dist(P, F_1) - dist(P, -F_2) = 2a - \pi\}.$$

Practical Application. These curves are used since more than 50 years in the LORAN System to determine the position of a ship on the ocean as follows. Consider a pair of radio stations which broadcast synchronized signals. If one measures at any

point P on the earth the time difference with which a pair of signals from the two stations arrives, then one knows the difference of the two distances from P to the radio stations. Therefore sea charts were prepared which show the curves of constant difference of the distances to the two radio stations. This has to be done for several pairs of radio stations. In araes of the ocean where the families of curves (for at least two pairs of radio stations) intersect reasonably transversal it is sufficient to measure two time differences, then a look on the sea chart will show the ship's position as the intersection point of two curves, two sperical hyperbolas. On the site
http://webhome.idirect.com/...
~jproc/hyperbolic/index.html or
~jproc/hyperbolic/lorc_hyperbola.jpg
this is explained by the following map:

Graphics: Orbiot Technologies AB



LOP - The locus of all positions where the observed time difference between the arrival of signals from two stations are constant.

The LOP forms a hyperbola which gives rise to the designation of Loran-C as a hyperbolic radionavigation system.

Elementary Construction, 3DXM-demo

Begin by drawing a circle of radius $2a$ around F_1 (called *Leitkreis* in German). Next, for every point C on this circle we find a point X on the spherical ellipse as follows:
Let M be the midpoint of the great circle segment from C to F_2 and let T be the great circle through M and perpendicular to that segment. In other words, T is the symmetry line between C and F_2. Finally we intersect T with the *Leitkreis* radius from F_1 to C in X. — Because we used the symmetry line T we have $dist(X, C) = dist(X, F_2)$ and therefore:

$$\begin{aligned} dist(X, F_1) + dist(X, F_2) &= dist(X, F_1) + dist(X, C) \\ &= dist(C, F_1) = 2a. \end{aligned}$$

It is easy to prove that the great circle T is tangent to the ellipse at the point$X$.

Connection with Elliptic Functions

We met a family of ellipses all having the same focal points ('confocal') and also the orthogonal family of confocal hyperbolas in the visualization of the complex function $z \to z + 1/z$. In the same way two orthogonal families of confocal spherical ellipses show up in the visualization of elliptic functions from *rectangular tori* to the Riemann sphere (choose in the Action Menu: *Show Image on Riemann Sphere* and in the View Menu: *Anaglyph Stereo Vision*). — Note that in the plane all such families of confocal ellipses and hyperbolas are essentially the same, they differ only in scale. On the sphere we get different families for different rectangular tori, i.e. for different quadrupels of focal points $\{F_1, F_2, -F_1, -F_2\}$.

An Equation for the Spherical Ellipse

Abbreviate $\alpha := dist(X, F_1)$, $\beta := dist(X, F_2)$. The definition of a spherical ellipse says:

$$\cos(2a) = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$
$$\text{with } \cos\alpha = \langle X, F_1\rangle, \ \cos\beta = \langle X, F_2\rangle.$$

We want to write the equation in terms of the scalar products which are linear in X. Therefore we replace

$\sin^2 = 1 - \cos^2$ to get:

$$(1 - \cos^2 \alpha)(1 - \cos^2 \beta) = (\cos \alpha \cos \beta - \cos(2a))^2$$

or

$$1 - \cos^2 \alpha - \cos^2 \beta = -2 \cos(2a) \cos \alpha \cos \beta + \cos^2(2a)$$

or, by replacing the cosines by the scalar products:

$$\begin{aligned} \sin^2(2a)\langle X, X\rangle - \langle X, F_1\rangle^2 - \langle X, F_2\rangle^2 = \\ - 2\cos(2a) \cdot \langle X, F_1\rangle \cdot \langle X, F_2\rangle. \end{aligned}$$

Observe that this is a homogenous quadratic equation in $X = (x, y, z)$. In other words: Our spherical ellipse is the intersection of the unit sphere with a quadratic cone whose vertex is at the midpoint of the sphere. So we get the surprisingly simple result: If one projects a spherical ellipse from the midpoint of the sphere onto some plane then one obtains a (planar) conic section.

H.K.