$(K = -1)$ - Surfaces from SGE solitons *

In 3DXM there are six explicit solutions $q(x,t)$ of the Sine-Gordon Equation (SGE), namely:
`One-Soliton, Two-Soliton, Three-Soliton, Four-Soliton, Kink, Breather.`
Each of them determines a pair of first and second fundamental forms

$$\mathrm{I} = dx^2 + dt^2 + 2\cos q \; dx\,dt, \quad \mathrm{II} = 2\sin q \; dx\,dt,$$

for which the Gauss-Codazzi integrability conditions are satisfied. This says:
There are parametrized surfaces in $\mathbb{R}^3$ with these first and second fundamental forms. They have Gauss curvature $K = -1$. The parameter lines are asymptote lines on these surfaces and x, t are arclengths on the parameter lines. For more details see:
`About Pseudospherical Surfaces` ,
available from the Documentation Menu.

Since pictures of surfaces, drawn with asymptote line parametrization, do not give a good 3D-impression of the surface, 3DXM uses instead curvature line parametrizations, i.e. the parameters $u = x + t, v = x - t$. One first sees one curvature line computed as solution of an ODE. Next

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

the family of orthogonal curvature lines is drawn. This determines the parametrized surface. It is then rendered according to the viewer's choice.

Recall the definition $\operatorname{cosec} := 1/\sin$.

The **One-Soliton** solution of SGE (parameter aa):
$q(x,t) = 4\arctan(\exp(\ \operatorname{cosec}(aa\cdot\pi)\cdot x - \operatorname{cotan}(aa\cdot\pi)\cdot t\ ))$.

The **Two-Soliton** solution of SGE (parameters aa, bb):
Define a constant B and functions $A1, A2$ first.
$B := (\cos(bb\,\pi) - \cos(aa\,\pi))/(\cos((aa-bb)\pi) - 1)$,
$A1(x,t) := \operatorname{cosec}(aa\,\pi)x - \operatorname{cotan}(aa\,\pi)t$,
$A2(x,t) := \operatorname{cosec}(BB\,\pi)x - \operatorname{cotan}(bb\,\pi)t$, then put:
$q(x,t) := \frac{4\arctan(\ B\exp(A2(x,t)) - \exp(A1(x,t))\ )}{1+\exp(\ A1(x,t)+A2(x,t)\ )}$.

The **Three-Soliton** solution of SGE (params aa, bb, cc):
Define three auxiliary functions E, F, H first.
$E(\xi, x, t) := \exp(\ \operatorname{cosec}(\xi\,\pi)x + \operatorname{cotan}(\xi\,\pi)t\ )$,
$F(\xi_1, \xi_2, x, t) := \frac{\cos(\xi_1\,\pi) - \cos(\xi_2\,\pi)}{\cos((\xi_2-\xi_1)\pi)-1} \cdot \frac{E(\xi_1,x,t)-E(\xi_2,x,t)}{1+E(\xi_1,x,t)\cdot E(\xi_2,x,t)}$,
$H(\xi_1, \xi_2, \xi_3, x, t) := \frac{\cos(\xi_1\,\pi) - \cos(\xi_2\,\pi)}{\cos((\xi_2-\xi_1)\pi)-1} \cdot \frac{F(\xi_1,\xi_2,x,t)-F(\xi_2,\xi_3,x,t)}{1+F(\xi_1,\xi_2,x,t)\cdot F(\xi_2,\xi_3,x,t)}$,

$$q(aa,bb,cc,x,t) := 4\arctan(\exp(\ \operatorname{cosec}(bb\,\pi)x + \operatorname{cotan}(bb\,\pi)t\ )) \bullet 4\arctan(H(aa,bb,cc,x,t)).$$

The **Four-Soliton** solution of SGE (params aa, bb, cc, dd):
Define functions E, F, H (as before) and J, K first.
$E(\xi, x, t) := \exp(\ \operatorname{cosec}(\xi\,\pi)x + \operatorname{cotan}(\xi\,\pi)t\ )$,
$F(\xi_1, \xi_2, x, t) := \frac{\cos(\xi_1\,\pi) - \cos(\xi_2\,\pi)}{\cos((\xi_2-\xi_1)\pi)-1} \cdot \frac{E(\xi_1,x,t)-E(\xi_2,x,t)}{1+E(\xi_1,x,t)\cdot E(\xi_2,x,t)}$,

$H(\xi_1,\xi_2,\xi_3,x,t) := \frac{\cos(\xi_1\,\pi)-\cos(\xi_2\,\pi)}{\cos((\xi_2-\xi_1)\pi)-1} \cdot \frac{F(\xi_1,\xi_2,x,t)-F(\xi_2,\xi_3,x,t)}{1+F(\xi_1,\xi_2,x,t)\cdot F(\xi_2,\xi_3,x,t)}$,

$J(\xi_1,\xi_2,\xi_3,x,t) := \frac{H(\xi_1,\xi_2,\xi_3,x,t)+E(\xi_2,x,t)}{1-H(\xi_1,\xi_2,\xi_3,x,t)\cdot E(\xi_2,x,t)}$,

$K(\xi_1,\xi_2,\xi_3,\xi_4,x,t) :=$

$$\frac{\cos(\xi_1\,\pi)-\cos(\xi_4\,\pi)}{\cos((\xi_4-\xi_1)\pi)-1} \cdot \frac{J(\xi_1,\xi_2,\xi_3,x,t)-J(\xi_2,\xi_3,\xi_4,x,t)}{1+J(\xi_1,\xi_2,\xi_3,x,t)\cdot J(\xi_2,\xi_3,\xi_4,x,t)},$$

Finally

$q(aa,bb,cc,dd,x,t) :=$

$$4\arctan(F(aa,bb,x,t))+4\arctan(K(aa,bb,cc,dd,x,t)).$$

The **Kink** (soliton) solution of SGE (parameter aa):
$q(x,t) := 4\arctan(\ (aa+\frac{1}{4\,aa})x+(aa-\frac{1}{4\,aa})t\ )$.

The **Breather** (soliton) solution of SGE (parameter aa):
Abbreviate $\omega := \sqrt{1-aa^2}$, then

$q(x,t) := 4\arctan(\ \frac{\sin(\omega\,t)}{\omega} \cdot \frac{aa}{\cosh(aa\,x)}\ )$.

Recall that each of these solutions determines a first and a second fundamental form which satisfy the Gauss-Codazzi integrability conditions. Each parameter line gives a space curve via an ODE which is determined by the fundamental forms. Because of the integrability conditions these space curves fit together to form a surface of Gauss curvature $K=-1$.

H.K.