Monkey Saddle, Torus, Dupin Cyclide *

The **Monkey Saddle** is a saddle shaped surface with *three* down valleys, allowing the two legs and the tail of the monkey to hang down. At its symmetry point both principal curvatures are 0, and, this umbilic point is the simplest singularity of a curvature line field. Choose in the Action Menu: `Add Principal Curvature Fields`; in `Wireframe Display` the parameter lines are omitted, the curvature line fields (or one of them) represent the surface.
Its *Parametrization* as graph of a function is

$$F_{Monkey}(u, v) = (aa \cdot v,\ bb \cdot u,\ cc \cdot (u^3 - 3uv^2)).$$

In Geometry the word **Torus** usually implies a surface of revolution; often a circle in the x-z-plane ist rotated around the z-axis. In 3DXM an ellipse with axes bb, cc is rotated, its midpoint rotates in the x-y-plane on a circle of radius aa. The following*Parametrization* is used:

$$F_{Torus}(u, v) = \begin{pmatrix} (aa + bb \cdot \cos u) \cos v \\ (aa + bb \cdot \cos u) \sin v \\ cc \cdot \sin u \end{pmatrix}$$

Note that the parameter lines are principal curvature lines, see Action Menu: `Add Principal Curvature Fields`.

The **Torus** is also visualized among the Implicit Surfaces, we derive its equation. In the x-z-plane we have two ellipses

* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

and we multiply their equations:

$$
\begin{aligned}
&((\frac{x-aa}{bb})^2+(\frac{z}{cc})^2-1)\cdot((\frac{x+aa}{bb})^2+(\frac{z}{cc})^2-1)\\
=&(\frac{x^2-aa^2}{bb^2})^2+2(\frac{x^2+aa^2}{bb^2})((\frac{z}{cc})^2-1)+((\frac{z}{cc})^2-1)^2.
\end{aligned}
$$

For the rotation around the z-axis we have to replace x by $r=\sqrt{x^2+y^2}$. The second expression avoids square roots.

Implicit Equation of the **Torus**:

$$f_{Torus}(\vec{x})=f(r,z)=0 \text{ with } r=\sqrt{x^2+y^2} \quad \text{and}$$

$$f(r,z):=$$

$$(\frac{r^2-aa^2}{bb^2})^2+2(\frac{r^2+aa^2}{bb^2})((\frac{z}{cc})^2-1)+((\frac{z}{cc})^2-1)^2.$$

The **Cyclides of Dupin** are obtained by inverting the above torus in a sphere. The sphere of inversion has its center $\vec{m}=(dd,0,ee)$ in the x-z-plane and has radius ff. The `Default Morph` moves the center closer to the torus. Note that inversions map curvature lines to curvature lines.

The Inversion: $\vec{x}\mapsto \mathrm{Inv}(\vec{x}):=\dfrac{ff^2(\vec{x}-\vec{m})}{|\vec{x}-\vec{m}|^2}+\vec{m}+\begin{pmatrix}0\\0\\hh\end{pmatrix}$,

Parametrization: $F_{Cyclide}(\vec{x}):=\mathrm{Inv}(F_{Torus}(\vec{x}))$,

Implicit Equation: $f_{Cyclide}(\vec{x}):=f_{Torus}(\mathrm{Inv}^{-1}(\vec{x}))=0$.