Möbius Strip and Klein Bottle *

Other non-orientable surfaces in 3DXM:
Cross-Cap, Steiner Surface, Boy Surfaces.

The **Möbius Strip** is the simplest of the non-orientable surfaces. On all others one can find Möbius Strips. In 3DXM we show a family with ff halftwists (non-orientable for odd ff, $ff = 1$ the standard strip). All of them are *ruled surfaces*, their lines rotate around a central circle. Möbius Strip *Parametrization:*

$$F_{Möbius}(u,v) = \begin{pmatrix} aa(\cos(v) + u\cos(ff\cdot v/2)\cos(v)) \\ aa(\sin(v) + u\cos(ff\cdot v/2)\sin(v)) \\ aa\,u\sin(ff\cdot v/2) \end{pmatrix}.$$

Try from the View Menu: `Distinguish Sides By Color`. You will see that the sides are not distinguished—because there is only one: follow the band around.

We construct a **Klein Bottle** by curving the rulings of the Möbius Strip into figure eight curves, see the Klein Bottle *Parametrization* below and its `Range Morph` in 3DXM.

$$w = ff\cdot v/2$$

$$F_{Klein}(u,v) = \begin{pmatrix} (aa + \cos w \sin u - \sin w \sin 2u)\cos v \\ (aa + \cos w \sin u - \sin w \sin 2u)\sin v \\ \sin w \sin u + \cos w \sin 2u \end{pmatrix}.$$

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

There are *three different Klein Bottles* which cannot be deformed into each other through immersions. The best known one has a reflectional symmetry and looks like a weird bottle. See formulas at the end. The other two are mirror images of each other. Along the central circle one of them is left-rotating the other right-rotating. See the `Default Morph` of the *Möbius Strip* or of the *Klein Bottle*: both morphs connect a left-rotating to a right-rotating surface.

On the *Boy Surface* one can see different Möbius Strips. The `Default Morph` begins with an equator band which is a Möbius Strip with *three halftwists*. As the strip widens during the deformation it first passes through the triple intersection point and at the end closes the surfaces with a disk around the center of the polar coordinates.
Moreover, each meridian is the centerline of an *ordinary Möbius Strip*. Our second morph, the `Range Morph`, rotates a meridian band around the polar center and covers the surface with embedded Möbius Strips. - We suggest to also view these morphs using `Distinguish Sides By Color` from the View Menu.

On the Steiner Surface and the Cross-Cap the Möbius Strips have self-intersections and are therefore more difficult to see. The `Default Morph` for the *Steiner Surface* emphasizes this unusual Möbius Strip. - The `Range Morph` of the *Cross-Cap* shows a family of embedded disks, except at the last moment, when opposite points of the boundary are identified, covering the self-intersection segment twice.

The construction of the mirror symmetric Klein bottle starts from a planar loop with parallel, touching ends and zero velocity at the ends (see the default morph):

$$
\begin{aligned}
cx(u) &= -cc \cdot \cos(u), & 0 \le u \le 2\pi \\
cy(u) &= ee \cdot \sin(u^3/\pi^2), & 0 \le u \le \pi \\
cy(u) &= ee \cdot \sin((2\pi - u)^3/\pi^2), & \pi \le u \le 2\pi \\
(nx, ny)(u) &\text{ is the unit normal,} & 0 < u \le 2\pi
\end{aligned}
$$

The Klein bottle is a tube of varying radius $\mathrm{rad}(u)$ around this curve. The function rad(u) is experimental:

$$\mathrm{rad}(u) = bb \cdot (dd + \sin(1.5\pi \cdot (1 - cos^3((u + 0.5)/2.85))));$$

The parameters cc, ee, bb just control the amplitudes of the functions; $dd > 1$ allows to vary the ratio between maximum and minimum of $\mathrm{rad}(u)$.

$$F_{symKlein}(u, v) = aa \cdot \begin{pmatrix} cx(u) + \mathrm{rad}(u) \cdot nx(u)\cos(v) \\ cy(u) + \mathrm{rad}(u) \cdot ny(u)\cos(v) \\ \mathrm{rad}(u) \cdot \sin(v) \end{pmatrix}.$$

H.K.