The Mandelbrot Set And Its Julia Sets *

If one wants to study iterations of functions or mappings, $f^{\circ n} = f \circ \cdots \circ f$, as n becomes arbitrarily large then *Julia sets* are an important tool. They show up as the boundaries of those sets of points p whose iteration sequences $f^{\circ n}(p)$ converge to a selected *fixed point* $p_f = f(p_f)$. One of the best studied cases is the study of iterations in the complex plane given by the family of quadratic maps

$$z \to f_c(z) := z^2 - c.$$

The *Mandelbrot set* will be defined as a set of parameter values c. It provides us with some classification of the different 'dynamical' behaviour of the functions f_c in the following sense: If one chooses a c-value from some specific part of the Mandelbrot set then one can predict rather well how the iteration sequences $z_{n+1} := f_c(z_n)$ behave.

1) Infinity is always an attractor. Or, more precisely, for each parameter value c we can define a

*This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

Radius $R_c \geq 1$ such that for $|z| > R_c$ the iteration sequences $f^{\circ n}(z)$ converge to infinity. Proof: The triangle inequality shows that $|f_c(z)| \geq |z|^2 - |c|$ and then $|f_c(z)| > |z|$ is certainly true if $|z^2| - |c| > |z|$. Therefore it is sufficient to define $R_c := 1/2 + \sqrt{1/4 + |c|}$, which is the positive solution of $R^2 - R - |c| = 0$.
This implies: if we start the iteration with $z_1 > R_c$ then the absolute values $|z_n|$ increase monotonically—and indeed faster and faster to infinity. Moreover, any starting value z_1 whose iteration sequence converges to infinity will end up after *finitely many iterations* in this neighborhood of infinity, $U_\infty := \{z \in \mathbb{C} \mid |z| > R_c\}$. The set of all points whose iteration sequence converges to infinity is therefore an open set, called the attractor basin $A_\infty(c)$ of infinity.

2) Definition of the Julia set J_c. On the other hand, the attractor basin of infinity is never all of $\mathbb{C}$, since f_c has fixed points $z_f = 1/2 \pm \sqrt{1/4 + c}$ (and also points of period n, that satisfy a polynomial equation of degree 2^n, namely $f^{\circ n}(z) = z$).

Definition. The nonempty, compact boundary of the attractor basin of infinity is called the *Julia set of f_c*,

$$J_c := \partial A_\infty(c).$$

Example. If $c = 0$ then the exterior of the unit circle is the attractor basin of infinity, its boundary, the unit circle, is the Julia set J_0. The open unit disk is the attractor basin of the fixed point 0 of f_c. The other fixed point 1 lies on the Julia set; 1 is an expanding fixed point since $f_c'(1) = 2$; its iterated preimages $-1, \pm\mathbf{i}, \ldots$ all lie on the Julia set.

Qualitatively this picture persists for parameter values c near 0 because the smaller fixed point remains attractive. However, the Julia set immediately stops being a smooth curve—it becomes a continuous curve that oscillates so wildly that no segment of it has finite length. Its image is one of those sets called a *fractal* for which a fractional dimension between 1 and 2 can be defined. Our rainbow coloration is intended to show J_c as a continuously parametrized curve. We next take a more carefull look at attractive fixed points.

3) c-values for which one fixed point of f_c is attractive.

There is a simple criterion for this: if the derivative at the fixed point satisfies $|f_c'(z_f)| = |2z_f| < 1$ then z_f is a linearly attractive fixed point; if $|2z_f| > 1$ then z_f is an expanding fixed point; if the derivative has absolute value 1 then no general statement is true (but interesting phenomena occur for special values of the derivative).

Since the sum of the two fixed points is 1, the derivative f_c' can have absolute value < 1 at most at one of them. Let w_c be that square root of $1+4c$ having a positive real part. Then $|1-w_c|$ is the smaller of the absolute values (of the derivatives of f_c at the fixed points). The set of parameter values c with a (linearly) attractive fixed point of f_c is therefore the set $\{c \mid \ |1-w_c| < 1\}$, or $\{c = (w^2-1)/4 \mid \ |1-w| < 1\}$. In other words, the numbers $1+4c$ are the squares of numbers w that lie in a disk of radius one with 0 on its boundary. The apple shaped boundary is therefore the square of a circle through 0. It is called a *cardioid*.

4) The definition of the Mandelbrot set in the parameter plane.

The behavior of the iteration sequence $z_{n+1} := f_c(z_n)$ in the z-plane depends strongly on the value of the parameter c. It turns out that for those c satisfying $|c| > R_c$, the set of points z whose iteration sequences do *not* converge to infinity has area $= 0$. Such points are too rare to be found by trial and error, but one can still compute many as iterated preimages of an unstable fixed point. It follows from $|c| > R_c$ that only the points of the Julia set J_c do not converge to infinity. Moreover, the Julia set is no longer a curve, but is a totally disconnected set: no two points of the Julia set can be joined by a curve inside the Julia set. (In this case our coloration of J_c has no significance.)

The Mandelbrot set is defined by the opposite behaviour of the Julia sets:

$$\text{Mandelbrot Set}: \mathbf{M} := \{c \mid \ J_c \ \text{ is a connected set}\}$$

There is an 80 year old theorem by Julia or Fatou that says:

$$\begin{aligned}\mathbf{M} &= \{c\ ;\ \ f_c^{\circ n}(0) \quad \text{stays bounded}\} \\ &= \{c\ ;\ \ |f_c^{\circ n}(0)| < R_c \ \text{ for all } n\}.\end{aligned}$$

This provides us with an algorithm for determining the complement of $\mathbf{M}$; namely $c \notin \mathbf{M}$ if and only if the iteration sequence $\{f_c^{\circ n}(0)\}$ reaches an absolute value $> R_c$ for some positive integer n. (But, the closer c is to $\mathbf{M}$, the larger this termination number n becomes). On the other hand, if f_c has an attractive fixed point, then it is also known that $\{f_c^{\circ n}(0)\}$ converges towards that fixed point. The interior of the cardioid described above is therefore part of the Mandelbrot set, and in fact it is a large part of it.

As *experiments* we suggest to choose c-values from the apple-shaped belly of the Mandelbrot set and observe how the Julia sets deform as c varies from 0 to the cardioid boundary. For an actual animation, choose the deformation interval with the mouse (Action Menu) and then select 'Morph' in the Animation Menu. To see how the derivative at the fixed point controls the iteration near the fixed point, choose 'Iterate Forward'

(Action Menu) and watch how chosen points converge to the fixed point. This is very different for c from different parts of the Mandelbrot belly.

5) Attractive periodic orbits. As introduction let us determine the orbits of period 2, i.e., the fixed points of $f_c \circ f_c$ that are not also fixed points of f_c. Observe that:

$$\begin{aligned} f_c \circ f_c(z) - z &= z^4 - 2cz^2 - z + c^2 - c \\ &= (z^2 - z - c)(z^2 + z - c + 1). \end{aligned}$$

The roots of the first quadratic factor are the fixed points of f_c, the roots of the other quadratic factor are a pair of points that are not fixed points of f_c, but are fixed points of $f_c \circ f_c$, which means, they are an orbit of period 2, clearly the only one. Such an orbit is (linearly) attractive if the product of the derivatives at the points of the orbit has absolute value < 1. The constant coefficient in the quadratic equation is the product of its roots, i.e. the product of the points of period 2 is $1-c$. Therefore: The set of c-values for which the orbit of period 2 is attractive is the disk $\{c\ ;\ \ |1-c| < 1/4\}$. Again, this disk is part of the Mandelbrot set since

$\{f_c^{\circ n}(0)\}$ has the two points of period 2 as its only limit points.

The interior of the Mandelbrot set has only two components that are explicitly computable. These are the c-values giving attractive fixed points or attractive orbits of period 2. For example, the points of period 3 are the zeros of a polynomial of degree 6, namely:

$$\begin{aligned} & \big(f_c \circ f_c \circ f_c(z) - z\big)/(z^2 - z - c) \\ = \quad & z^6 + z^5 + (1-3c)z^4 + (1-2c)z^3 + \\ & +(1-3c+3c^2)z^2 + (c-1)^2 z + 1 - c(c-1)^2. \end{aligned}$$

But since this polynomial cannot be factored (with c a parameter) into two polynomials of degree 3 it does not provide us with a description of the attractive orbits of period 3. However, it does give those c-values for which the period 3 orbits are superattractive (i.e. $(f^{\circ 3})'(orbit\,point) = 0$), since in this case the constant term must vanish. Approximate solutions of $1-c(c-1)^2 = 0$ are $c = 1.7549, c = 0.12256 \pm 0.74486i$. One can navigate the Mandelbrot set and observe that the complex solutions are between the two biggest blobs that touch the primary apple from either side.

Linearly attractive orbits always have c-values which belong to open subsets of the Mandelbrot set (in particular all the blobs touching the two explicit components), but the closure of these open subsets does not exhaust the Mandelbrot set. For example for $c = i$ the orbit of 0 is $0 \mapsto -i \mapsto -1 - i \mapsto i \mapsto -1 - i \ldots$, i.e., after two preliminary steps it reaches an orbit of period 2. Since this orbit stays clearly bounded we have $i \in \mathbf{M}$ (by the criterium quoted before). On the other hand, if the iteration $z \mapsto z^2 - i$ had any attractor (besides ∞), then the orbit of 0 would have to converge to the attracting orbit. Therefore there is no attractor and no attractor basin. In fact, the complement of the Julia set is the (simply connected) attractor basin of ∞. Because of its appearance, this Julia set is called a dendrite.

To generalize this observation, consider, for any c, the orbit of 0: $0 \mapsto -c \mapsto c^2 - c \mapsto c^4 - 2c^3 + c^2 - c \mapsto (c^4 - 2c^3 + c^2 - c)^2 - c \mapsto \ldots$. If 0 is on a periodic orbit for some c, then this orbit is superattractive. If the periodicity starts later then this periodic orbit may not be an attractor even though the orbit of 0 reaches it in finitely many steps. For example $c^2 - c$ is periodic of period 3, if $c^3 \cdot (c-2) \cdot (c^3 - 2c^2 + c - 1)^2 \cdot (c^6 - 2c^5 + 2c^4 - 2c^3 + c^2 + 1) = 0$; $c = 2$ is the largest point on the Mandelbrot set, the third factor has as roots the three c-values (mentioned before) for which the iteration has superattractive orbits of period 3. The last factor has the root $c = 1.239225555 + 0.4126021816 \cdot i$, its Julia set is another dendrite. A third dendrite is obtained, for example, if the 4^{th} point $c^4 - 2c^3 + c^2 - c$ in the orbit of 0 is a fixed point, which is the case if $c^4(c-2)(c^3 - 2c^2 + 2c^2 - 2) = 0$; here the last factor has the numerical solutions $c = 1.543689$ and $c = 0.2281555 \pm 1.1151425 \cdot i$.

6) Suggestions for experiments. The final entry in the Action Menu for the Julia set fractal is a hierachical menu with five submenus, each of which lists a number of related c-values that you may select. The c-values in these menus were selected because they typify either some special topological property of the associated Julia set or some dynamical property of the iteration dynamics of $z \mapsto z^2 - c$, and these properties are referenced by special abbreviations added to the menu item. (In addition some menu items also list a “name” that is in common use to refer to the Julia set, usually deriving from its shape). For convenience we will list in the next couple of pages all the items from these five menus, but first we explain the abbreviations used to describe them.

Abbreviations used in the following lists of interesting C-values. ‘FP’ means ‘fixed point’, the corresponding c-values are from the belly of the Mandelbrot set. ‘cyc k’ means ‘cyclic of period k’, the corresponding c-values are from the blobs directly attached to the belly; its Julia sets have a fixed point which is a common boundary point of k components of the attractor basin and the attractive orbit wanders cyclicly through these k components. ‘per $2 \cdot 3$’ means: this c-value has an attractor of period 6 and the c-value is from a blob which is attached to the disk in $\mathbf{M}$ (which gives the attractive orbits of period 2). By contrast, ‘per $3 \cdot 2$’ means that the c-value is from the biggest blob which is attached to a period-3 blob (attached to the belly); its attractor has also period 6, but the open sets through which the attractive orbit travels are arranged quite differently in the two cases. One should compare both of them with the cyclic attractors of period 2 resp. 3. The abbreviation 'tch 1-2' means that the c-value is in the Mandelbrot set a common boundary point between the belly (i.e. the component of attractive fixed points) and the component of attractors of period 2. For the 'Siegel disks' see Nr. 8 of this ATO first; the column entry in the list gives the rotation number of the derivative (of the iteration map) at the fixed point. In the dendrite section of the list we mean by 'ev per 2' that the orbit of 0 is 'eventually periodic with period 2', as explained in Nr5 of this ATO. Finally, if $c \notin \mathbf{M}$ then the Julia set is a totally disconnected Cantor set and there are no such easy distinctions between different kinds of behaviour of the iteration on the Julia set (all other points are iterated to ∞).

Interesting C-values From the Action Submenus.

$C-$values				Popular Name	Behaviour
Attractors Menu.					
0.0	+	0.0	$\cdot i$	Circle	FP
0.0	+	0.1	$\cdot i$	Rough Circle	FP
0.127	+	0.6435	$\cdot i$	Near-Rabbit	FP
−0.353	−	0.1025	$\cdot i$	Near-Dragon	FP
0.7455	+	0.0	$\cdot i$	Near San Marco	FP
1.0	+	0.0	$\cdot i$		cyc 2
1.0	+	0.2	$\cdot i$		cyc 2
0.1227	+	0.7545	$\cdot i$	Rabbit	cyc 3
1.756	+	0.0	$\cdot i$	Airplane	cyc 3
−0.2818	+	0.5341	$\cdot i$		cyc 4
1.3136	+	0.0	$\cdot i$		per $2 \cdot 2$
−0.3795	+	0.3386	$\cdot i$		cyc 5
0.5045	+	0.5659	$\cdot i$		cyc 5
−0.3909	+	0.2159	$\cdot i$		cyc 6
0.1136	+	0.8636	$\cdot i$		per $3 \cdot 2$
1.1409	+	0.2409	$\cdot i$	Rabbit's Shadow	per $2 \cdot 3$
−0.3773	+	0.1455	$\cdot i$		cyc 7
−0.1205	+	0.6114	$\cdot i$		cyc 7
−0.36	−	0.1	$\cdot i$	Dragon	cyc 8
0.3614	+	0.6182	$\cdot i$		cyc 8
−0.3273	+	0.5659	$\cdot i$		per $4 \cdot 2$
1.0	+	0.2659	$\cdot i$		per $2 \cdot 4$
1.3795	+	0.0	$\cdot i$		per $2 \cdot 2 \cdot 2$
0.0318	+	0.7932	$\cdot i$	Rabbit Triplets	per $3 \cdot 3$
−0.0500	+	0.6318	$\cdot i$		cyc 10
−0.4068	+	0.3409	$\cdot i$		per $5 \cdot 2$
0.5341	+	0.6023	$\cdot i$		per $5 \cdot 2$
0.9205	+	0.2477	$\cdot i$		per $2 \cdot 5$
1.2114	+	0.1545	$\cdot i$		per $2 \cdot 5$
0.6977	+	0.2818	$\cdot i$		cyc 11
0.4864	+	0.6023	$\cdot i$	Quintuple Rabbits	per $5 \cdot 3$
0.65842566307252	−	0.44980525145595	$\cdot i$	SuperAttractor	per 21

Interesting C-values From the Action Submenus.

C−values				Popular Name	Behaviour
Between Attractors Menu.					
0.75	+	0.0	$\cdot i$	San Marco	tch 1-2
1.25	+	0.0	$\cdot i$	San Marco's Shadow	tch 2-2·2
0.125	+	0.64952	$\cdot i$	Balloon Rabbit	tch 1-3
−0.35676	+	0.32858	$\cdot i$		tch 1-5
Siegel Disks Menu.					
0.390540870218	+	0.586787907347	$\cdot i$		$2\pi \cdot i \cdot$ gold
−0.08142637539	+	0.61027336571	$\cdot i$		$2\pi \cdot i/\sqrt{2}$
0.66973645476	−	0.316746426417	$\cdot i$		$2\pi \cdot i/\sqrt{5}$
One Simply Connected Open Component Menu.					
0.0	+	1.0	$\cdot i$	Dendrite	ev per 2
0.2281554936539	+	1.1151425080399	$\cdot i$	Dendrite	FP[after 3]
1.2392255553895	−	0.4126021816020	$\cdot i$	Dendrite	ev per 3
−0.4245127190500	−	0.2075302281667	$\cdot i$		FP after 7
1.1623415998840	+	0.2923689338965	$\cdot i$		per 2 after 7
Outside Mandelbrot set Menu.					
0.765	+	0.12	$\cdot i$	Cantor set	
−0.4	−	0.25	$\cdot i$	Cantor set	
−0.4253	−	0.2078	$\cdot i$	Cantor set	

An *experiment* which one should always make after one has computed a Julia set for some c from the Mandelbrot set: Remember from which part of $\mathbf{M}$ c came and then 'Iterate Forward' (Action Menu) mouse selected points until they visually converge to a periodic attractor. Observe how the shape of the Julia set lets one guess the period of its attractor and how this relates to the position of c in $\mathbf{M}$.

7) Computation of the Julia set. In addition to the attractor at infinity there is at most one further attractor in the $z \to (z^2 - c)$ systems. All preimages of non-attractive fixed points or non-attractive periodic orbits are points on the Julia set. Since $|f_c'| > 1$ along the Julia set (with some exceptions), the preimage computation is numerically stable. This is a common method for computing Julia sets.

In our program we compute preimages starting from the circle $\{z;\ |z| = R_c\}$ around the wanted Julia set. Under inverse images these curves converge from outside to the Julia set. Such an approximation by curves allows us to color the Julia set in a continuous way and thus emphasize that, despite its wild looks it is the image of a continuous curve—at least for $c \in M$, otherwise we recall that the Julia set is totally disconnected, so in particular is not the image of a curve. Our computation works also for $c \notin \mathbf{M}$, since our 'curves' of course consist of only finitely many points, and the inverse images of each of these points have their limit points on the Julia set.

8) Self-similarity of a Julia set. A well advertised property of these Julia sets is their so called 'self-similarity'. By this one means: Take a small piece of the Julia set and enlarge it; the result looks very much like a larger piece of that same Julia set. For the Julia sets of the present quadratic iterations, this self-similarity is easily understood from the definitions: The iteration map f_C is a *conformal* map that stretches its Julia set 1:2 onto itsself. In other words, the iteration map itself maps any small piece of its Julia set to roughly twice as large a piece, and it does so in an angle preserving way. From this point of view self-similarity should come as no surprise.

9) Siegel Disks. We next would like to explain an experimentally observable phenomenon that mathematicians find truly surprising, but this needs a little preparation.

Simplifying Mappings. Imagine that we want to describe something on the surface of the earth, for example a walk. For a long time, people have been more comfortable giving the description on a map of the earth rather than on the earth itself. Mathematicians view a map of the earth more precisely as a mapping F from the earth to a piece of paper and they describe (or even prove) properties of the map by properties of the mapping F. An example of a useful property is 'conformality': angles between curves on the earth are the same as the angles between the corresponding curves on the map.

Conjugation by simplifying mappings. Let us consider one of the above iteration maps f_C and assume that it has an attractive fixed point z_f with derivative $q := f_C'(z_f),\ |q| < 1$. The simplest map with the same derivative is the linear map $L(z) := q \cdot z$. It is the definition of derivative that the behaviour of f_C near the fixed point looks 'almost' like the behaviour of L near its fixed point 0, and 'almost' means: the smaller the neighborhoods of the fixed points (on which the maps are compared) the more the maps look alike. But more is true for f_c because of the assumption $|q| < 1$, we have the *theorem*: There exists on a fixed(!) neighborhood of the fixed point z_f a simplifying map F to a neighborhood of $0 \in \mathbb{C}$ that makes f_c look *exactly* like its linear approximation L, by which we mean: $f_c = F^{-1} \circ L \circ F$. In particular, this tells us everything about the iterations of f_c in terms of the iterations of L because they also look the same when compared using ('conjugation' by) F: $\quad f_c^{\circ n} = F^{-1} \circ L^{\circ n} \circ F$.

Siegel's Theorem. The previous result cannot be true in general if $|q| = 1$. For example if $q = \exp(2\pi i/k)$, then $L^{\circ k} = \mathrm{id}$, but $f_c^{\circ k} \neq \mathrm{id}$. Therefore they cannot look alike under a simplifying (i.e., 'conjugating') mapping F. But if $z \to q \cdot z$ is an irrational rotation and if some further condition is satisfied, for example if $q := \exp(2\pi i/\sqrt{2})$, then there is again such a simplifying mapping F such that f_c looks near that fixed point exactly like its linearization, namely: $f_c = F^{-1} \circ L \circ F$.

Experiment. While Siegel's proof insures only *very* small neighborhoods on which the simplifying mapping F exists, these neighborhoods are surprisingly large in the present case. One can 'observe' Siegel's theorem by first choosing $c = ((1-q)^2-1)/4$ such that $f_c'(z_f) = q$ with $q = \exp(2\pi i \cdot k/\sqrt{p})$, p prime (or square free), then one chooses points on a fairly straight radial curve from the fixed point almost out to the Julia set. Under repeated iterations these points travel on closed curves around the fixed point ('circles' when viewed with F) and all of them travel with the same angular velocity, i.e., one observes that they remain on non-intersecting radial curves.

H.K.