Loxodrome*

A Loxodrome (aka a rhumb line) is a route that a boat would take if it kept a constant compass heading (so that on a Mercator projection it is simply a straight line). To be more formal, a loxodrome is a path on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ that makes a constant angle with the great circles of longitude ("meridians"). Recall that the logarithmic spirals in the (complex) plane $\mathbb{C}$ make a constant angle with the rays through the origin. *Stereographic projection* $\mathrm{St}\colon \mathbb{C} \mapsto \mathbb{S}^2$ unites these two facts: It maps the radial lines in $\mathbb{C}$ to the meridians on $\mathbb{S}^2$ and it is also angle preserving ("conformal"). The logarithmic spirals - which meet the radial lines in $\mathbb{C}$ under constant angles - are therefore mapped by stereographic projection to the loxodromes on $\mathbb{S}^2$ - which meet the meridians under constant angles.
On the other hand, the complex exponential $\exp : \mathbb{C} \mapsto \mathbb{C}$ is also conformal and it maps the lines parallel to the real axis to the radial lines in $\mathbb{C}$ and all other straight lines to curves which meet the radial lines under constant angles, i.e. the logarithmic spirals. These are therefore parametrized as $c(t) = \exp((1 + i \cdot aa) \cdot t)$ and the loxodromes on the sphere are given parametrically as

$$t \mapsto \mathrm{St}(\exp((1 + i \cdot aa) \cdot t)).$$

1: Their osculating circles all lie on the sphere. And 2: the Mercator map from the sphere is: $\exp^{-1} \circ \mathrm{St}^{-1} : \mathbb{S}^2 \mapsto \mathbb{C}$.

R.S.P.

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/