

About Hilbert's Square Filling Curve*

See also: Koch Snowflake, Dragon Curve
Speed up demos by pressing DELETE

In 1890—the year the German Mathematical Society was founded, David Hilbert published a construction of a continuous curve whose image completely fills a square. At the time, this was a contribution to the understanding of continuity, a notion that had become important for Analysis in the second half of the 19th century. Today, Hilbert's curve has become well-known for a very different reason—every computer science student learns about it because the algorithm has proved useful in image compression. In this application one has to enumerate a first square, its four half size subsquares, their sixteen quarter-size subsquares and so on, in such a way that squares whose numbers are close are also close to each other geometrically. In other words, the continuity of this space filling curve is now important, in contrast to the fact that the curve was considered a pathological example of continuity for many years after Hilbert's discovery.

It was known in 1890 that such a curve, i.e., a continuous map c of $[0, 1]$ onto $[0, 1] \times [0, 1]$, could not be one-to-one, i.e., *Certain pairs of points t_1, t_2 of the interval $[0, 1]$ must have the same image $c(t_1) = c(t_2) \in [0, 1] \times [0, 1]$.* This

* This file is from the 3D-XplorMath project. Please see:

<http://3D-XplorMath.org/>

led Hilbert to give a special twist to his construction: He gave a sequence of polygon approximations of the strange limit curve that, surprisingly, were all one-to-one! In retrospect it seems almost as if Hilbert foresaw what would be needed a century later in image compression; when people say that they are using Hilbert's square-filling curve, they mean more precisely that they are using Hilbert's approximations to that curve!

The basis of Hilbert's construction is a single step that is repeated over and over again. We first explain a simplified version, although this does not exactly give Hilbert's one-to-one approximations that made the construction so famous. Assume that we already have a curve inside the square and joining the left bottom corner to the right bottom corner. 3DXM offers four different initial such curves, leading to quite different pictures. The basic construction step is to scale the square and its curve by $\frac{1}{2}$ and put **four** copies of this smaller square side by side in the original square, in such a way that these four smaller copies of the curve fit together to form a new curve from the left bottom corner to the right bottom corner of the original square. But instead of reading more words, we suggest that you view the default approximations of the Hilbert curve in 3DXM. We use a rainbow coloration to emphasize the continuous parametrization, and we repeat the colors **four** times to emphasize that four copies of the previous approximation make up the new one.

The two end points of the curve (to which this basic iter-

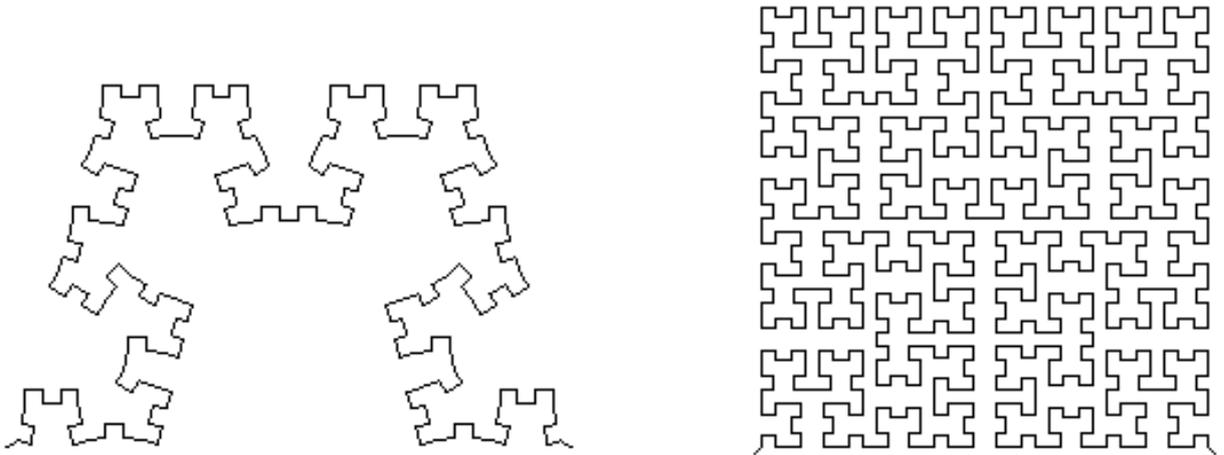
ation step is applied) play a special role, on the one hand they lead at each iteration step to more points that are already points on the limit curve, on the other hand exactly these easy points lead to double points on the approximations! Hilbert therefore removed small portions of the curve near its two end points before he applied the above iteration step. One can see how these Hilbert approximations manage to stay one-to-one and how they wander through all the little squares of the current subdivision of the original square—and these are just the properties used in image compression.

In 3DXM one can choose with the parameter cc between several initial curves. An even number and the following odd number choose the same curve, but for even cc the Hilbert iteration is done *without* the endpoints and for odd cc *including* the endpoints. In the Action Menu one can switch between Hilbert’s approximation ($cc=0$) and one that emphasizes the iteration of the endpoints ($cc=5$).

Finally we add to the above descriptive part some more technical explanations, namely how to understand the limit as a continuous curve. Select the Action Menu entry “Emphasize Limit Points”. The first shown step (for our default value $cc = 5$) is a curve that is mostly a straight segment, but has also two little wiggles, that emphasize the initial point $c(0)$ and the end point $c(1)$. The second step is a curve with four straight segments that join five wiggles, the points $c(j/4)$, $j := 0, \dots, 4$. These points are really points on the limit curve because they remain fixed

under all further applications of Hilbert's basic construction step. In the third step we get 17 wiggles, the points $c(j/16)$, $j := 0, \dots, 16$ of the limit curve, and so on. The 3DXM demo shows six such iterations. One can deduce from this the continuity of the limit curve if one proves for these approximations:

$$|t_2 - t_1| \leq \frac{1}{4^n} \Rightarrow |c(t_2) - c(t_1)| \leq \frac{1}{2^n}.$$



Early iterations for $bb = 0.4$ (left), for $bb = 0.5$ (right, the Hilbert case).

H.K.