Fractals & Chaos

The mathematical ideas that are encapsulated in the terms "Fractal" and "Chaos" are not easy to make precise in non-technical terms. Yet somehow these terms have become "buzz words" that have caused an unprecedented stir in the popular culture. Usually bookstores have very few mathematics books directed at the layman, but the past decade has seen an explosion of popular books with the stated goals of explaining the theory and applications of fractals and chaos to the non-mathematician.

These concepts have also proved exciting for professional mathematicians. The beginnings of both theories can be traced back at least a century, but in retrospect it is clear that each required the visualization capabilities that modern computers have provided in order to progress beyond a fairly primitive stage. In fact both areas were dormant for many years prior to the 1970s when the ability to model fractals and chaotic systems on computers provided some striking examples that gave the necessary impetus for important progress.

We will not try here to give precise definitions of what it means to say that a geometric object is a fractal or that a dynamical system is chaotic. Rather, we will try to explain these concepts in intuitive terms that are directly related to the visualizations that 3D-XplorMath provides.

The first property is a geometric one that applies to subsets of the plane or higher dimensional spaces and goes by the name of scaling invariance. What this means is that there are arbitrarily small pieces of the set that when "scaled-up" (i.e., seen under a high power microscope) look very similar to much larger pieces of the set. Such sets are called *fractals*, because in a certain precise sense they have a dimension which is a non-integer value.

Chaos on the other hand describes a characteristic behavior of certain dynamical processes that is usually referred to as "sensitive dependence on initial conditions". For simplicity we will restrict our attention here to fairly special dynamical systems, namely discrete dynamical systems in the complex plane defined by a polynomial function, $f(z)$, of degree d greater than one. (In fact, the case of a quadratic polynomial already exhibits most of the interesting behavior.) Let us denote by $f^{\circ n}$ the polynomial obtained by composing f with itself n times. Roughly speaking, "complex dynamics" is concerned with the study of the asymptotic properties of the sequences of iterates $f^{\circ n}(z)$ as n teds to infinity, and how this asymptotic behavior changes as the initial condition, z, varies.

We define $B_\infty(f)$ to be the so-called attractor basin of infinity, namely the set of all initial conditions z for which the sequence $f^{\circ n}(z)$ tends to infinity. It is easy to see that $B_\infty(f)$ is an open set. Moreover, from $d \geq 2$ it follows that there is some disk, say of radius R, and a constant $K > 1$ such that $|f(z)| > K|z|$ for all initial conditions z with $|z| > R$, and it follows that the complement of the disk of radius R is included in $B_\infty(f)$, so in particular $B_\infty(f)$ is non-empty. On the other hand B_∞ cannot be the whole complex plane. For example, any solution of the polynomial equation $f(z) - z = 0$ is a fixed point of f so that $f^{\circ n}(z) = z$ for all n, and so z is not in B_∞. It follows that the boundary ∂B_∞ of the basin of infinity is a non-empty compact set, $J(f)$ called the Julia set of f.

Note that by definition $J(f)$ is a “bifurcation set” for f, in the sense that given any point j of $J(f)$ there are points p and q arbitrarily close to j such that $f^{\circ n}(p)$ tends to infinity while $f^{\circ n}(q)$ remains bounded. In other words, for z near j, the asymptotic behavior of the sequence $f^{\circ n}(z)$ depends sensitively on the initial value z, the hallmark of Chaos.

Example. If $f(z) = z^2$, show that $B_\infty(f)$ is the complement of the unit disk, so that $J(f)$ is the unit circle.

It turns out that this example is highly atypical—in general $J(f)$ is **not** a smooth curve, and in fact it is usually a fractal set, establishing a relation between fractals and chaos. For more details see the ATO for the Mandelbrot set and Julia sets.