Four Triply Periodic Minimal Surfaces

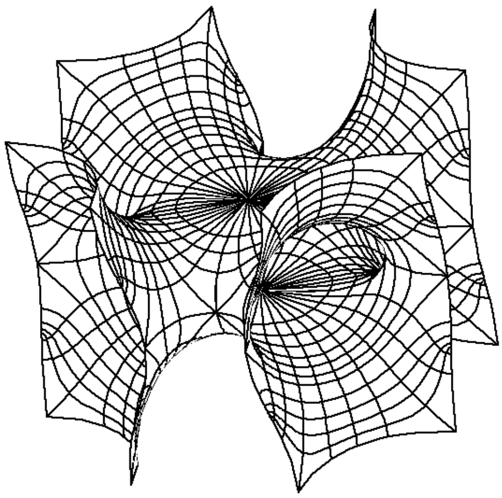
The four surfaces are:
· Neovius' surface inside a cube (no deformations)
· A. Schoen's S-S-surface in a quadratic prism (deforms)
· A. Schoen's H-T-surface in a hexagonal prism (deforms)
· A. Schoen's T-W-surface in a hexagonal prism (deforms)

The images in 3D-XplorMath show a fundamental domain for the group of translational symmetries. The curves on the boundary are planar symmetry arcs, reflection in their planes extends the fundamental domain to neighboring pieces. These planes, or rather their halfspaces that contain the fundamental domain, intersect in a crystallographic cell.

Neovius was a student of H. A. Schwarz. A. Schoen worked for NASA and he even made models of the surfaces he discovered. The T-W-surface is not in his list, but made in his spirit. Initially his descriptions were not accepted by mathematicians. In [Ka] it was shown that their existence follows easily with the conjugate Plateau construction (see 'About Minimal Surfaces') and Schoen was finally given credit for his discoveries.
What else should one observe in these pictures?
The images were not computed via the conjugate Plateau construction, but by using the Weierstrass representation. This representation requires the understanding of functions which are related to these minimal surfaces. To construct such functions is a mathematical challenge and to visualize them is not easy either. In 3D-XplorMath com-

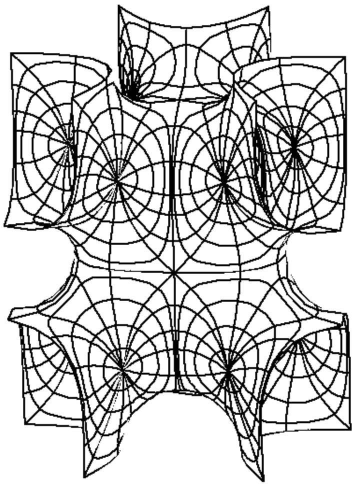
plex functions are (mostly) visualized by showing a grid in the domain and the image grid. We need functions that are well defined on the whole minimal surface. Thus visualization seems to require that we put a grid on the whole surface - which is impossible.

One such function is always the Gauss map which one can sort of 'see' by translating the normals of the surface to the origin, thus mapping the surface to the Riemann sphere. The four surface pictures are covered by patches which look like polar coordinates. This offers a different kind of visualization: The function on the surface which is responsible for these polar grids, maps the surface in such a way to the Riemann sphere that our grid becomes the *standard polar grid on the sphere.* One may compare this with the interpretation of a hiking map with level lines.

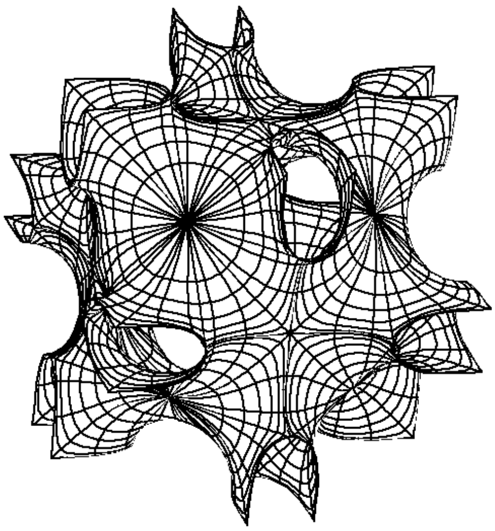


This S-S-surface shows these level lines for the Gauss map: The two big polar centers are a zero and a pole (of order 3); the preimage of a hemisphere (around 0 or ∞) in most directions ends at a horizontal symmetry line – i.e. at the preimage of the equator; but these horizontal arcs are connected by vertical (boundary) symmetry arcs, how is that possible? These vertical symmetry arcs have inflection points in their middle, i.e. double points of the Gauss map; the vertical symmetry arcs are therefore preimages of slits in the upper (resp. lower) hemisphere.

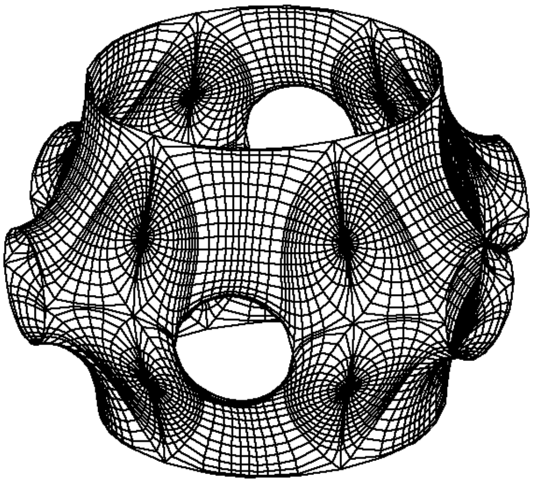
In the other three cases the level lines belong to functions which are obtained as quotient by a smmetry group. Although such a construction of a function is rather abstract, the grid lines at least show *how* the surface is mapped to the sphere.



Consider the T-W-surface first. Clearly there is a 120° rotation symmetry and orthogonal to its axis are three 180° rotation axes. Identification with this order 6 symmetry group makes out of the twelve polar grid patches just two, one around the northpole the other around the southpole of the quotient sphere. In other words: the parameter lines on this surface are the level lines of the quotient function to the sphere.



The Neovius surface has its eight polar centers on the space diagonals of its cubical crystallographic cell. The group of 180° rotations around the coordinate axes has order four. Identification by this symmetry group (modulo translations) makes again a sphere out of the Neovius surface and the 8 polar patches are the preimages under the quotient map of standard polar coordinates on the quotient sphere. The preimage of the equator consists, for each polar patch, of the twelve symmetry arcs which form the boundary of these polar patches. Our parameter lines, therefore, give an excellent impression of this quotient function.



Finally the H-T-surface. The fundamental piece itsself has a 60° symmetry rotation group. But one can see that the neighboring polar centers along the top boundary should be preimages of a northern and a southern hemisphere, while the polar centers which lie vertically above each other should be identified because there is a branch point of the identification map between them. We therefore take the symmetry group as for the T-W-surface: the 120° rotations around the vertical axis plus the 180° rotations around the horizontal axis through the branch points between the polar centers. Then again our parameter lines are the preimages of standard polar coordinates on the quotient sphere.

It still takes more effort to understand the Weierstrass representation. But if these images help to see functions from these minimal surfaces to the Riemann sphere then they have prepared the ground for the final step.

Bibliography

[Ka] Karcher, H.: The triply Periodic Minimal Surfaces of Alan Schoen And Their Constant Mean Curvature Companions. Manuscripta Math. 64, 291-357 (1989).

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