About the Classical Enneper Surface and some Polynomial Relatives

See also: About Minimal Surfaces

Definition with explicit formulas

The *classical Enneper Surface* is a minimal immersion of the complex plane, $\mathbb{C}$, into Euclidean space $\mathbb{R}^3$. It is given by the formula

$$F(z) := \mathrm{real}(\{z^3/3 - z,\ i * (z^3/3 + z),\ z^2\}).$$

If one wants to see coordinate lines on the image one can use Cartesian coordinates for the complex plane, $z := x + i \cdot y$, or polar coordinates $z := r \cdot (\cos\varphi + i \cdot \sin\varphi)$, and map those grids with F. The Cartesian choice is natural here since its parameter lines are principal curvature lines. However the polar choice also has merits—namely, rotations around the origin are isometries and the coordinate lines are orbits. Moreover all symmetry lines of the surface are radial parameter lines. The Action Menu of 3DXM allows to switch between these parametrizations.

In 3DXM one can also deform this classical surface, but we need to explain the significance of what one sees. See the last page of this text.

Early in the second half of the nineteenth century the *Enneper-Weierstraß representation* of minimal surfaces was discovered. Its main advantage is that it permits one to write a formula for a minimal surface in terms of important geometric quantities. Every surface in $\mathbb{R}^3$ can be mapped to the 2-sphere $\mathbb{S}^2$ by sending each point on the surface to the unit normal at this point; this map is called the geometric Gauß map N. For minimal surfaces this map is angle preserving, but orientation reversing. Composition of N with the orientation reversing stereographic projection therefore gives a map g from the surface into $\mathbb{C}$ which is both *and* orientation preserving. Finally, if we interpret $90°$ rotation on each tangent space of a surface in $\mathbb{R}^3$ as multiplication of tangent vectors by i, then with this convention g becomes a meromorphic function, the meromorphic Gauß map of the minimal surface. This meromorphic Gauß map is one-half of the *Weierstraß data* which are needed to write down the Enneper-Weierstraß representation. The remaining part of these data is the differential dF^3 of the third component of F, i.e., of the height function on the minimal surface. It might seem at first that we must know a minimal surface rather well before we have its Weierstraß data. However, on a large class of geometrically important minimal surfaces the situation is simple indeed. If a minimal surface is complete and has finite total curvature then the Gauß map g is determined—up to a constant factor—by its zeros and poles, in other words by its vertical normals. This important result extends to differentials, in particular to the differential of the height function, after we perform a small trick, namely extend the real valued differential dF^3 to a complex valued one by putting for every tangent vector v of the surface

$$\begin{aligned} dh(v) &:= dF^3(v) - i \cdot dF^3(i \cdot v) \\ &:= dF^3(v) - i \cdot dF^3(\mathrm{Rot}^{90}(v)). \end{aligned}$$

To make matters even simpler, observe that the points on the surface, where the normal is vertical, are the same points where the differential of the height function is zero. More precisely, the zeros and poles of g on the minimal surface are precisely the zeros of dh, even with the same multiplicity. (To complete this discussion we would have to study the situation at infinity, but we will omit this.) The main point is to point out that very few, *finitely many*, data about such minimal surfaces suffice to find their Weierstraß data and therefore explicitly parametrize them. Here is this famous formula:

Weierstraß Representation in terms of g, dh:

$$F(z) := \mathrm{Re}\left(\int_*^z \left\{\frac{1}{2}\left(g - \frac{1}{g}\right) dh, \frac{i}{2}\left(g + \frac{1}{g}\right) dh,\ dh\right\}\right)$$

The classical Enneper surface is obtained if we put $g(z) = z,\ dh = zdz$.
This generalizes to polynomials $P(z)$, put:
$g(z) = P(z),\ dh = P(z)dz$.

3DXM allows $P(z) := aa \cdot z + bb \cdot z^2 + cc \cdot z^3$.

The pure powers have again the rotations around the origin as metric isometry group and *polar coordinates* provide a much better view of these surfaces and are recommended. All surfaces of the associate family are, for $g(z) = z^k$, congruent. There are straight lines on the surface, and if one looks in the direction of the z-axis onto the surface, then the portion below these lines is drawn first. The default morph deforms two such surfaces into each other.