# On Curves Given By Their Support Function

This note is about smooth, closed, convex curves in the plane and how to define them in terms of their so-called *Minkowski support function* $h$. For quick reference we first show how, in 3D-XplorMath, $h$ can be modified by specifying parameters. Then we begin with a more general class of geometric objects, namely *convex bodies*.

## 1. Parameter Dependent Formulas

In 3D-XplorMath, the support function $h$ is given in terms of Fourier summands:

$$\begin{aligned} h(\varphi) := \;& aa + bb\cos(\varphi) + cc\cos(2\varphi) + \\ & dd\cos(3\varphi) + ee\cos(4\varphi) + ff\cos(5\varphi). \end{aligned}$$

In terms of this function we define the following curve:

$$c(\varphi) := h(\varphi) \cdot \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} + h'(\varphi) \cdot \begin{pmatrix} -\sin(\varphi) \\ +\cos(\varphi) \end{pmatrix}.$$

Differentiation shows that $c$ is given in terms of its unit normal and tangent vectors and the function $h$:

$$c'(\varphi) = (h + h'')(\varphi) \cdot \begin{pmatrix} -\sin(\varphi) \\ +\cos(\varphi) \end{pmatrix}.$$

One obtains curves with nonsingular parametrization ($|c'| > 0$) if $aa$ is choosen large enough. And since $h(\varphi)$ equals the scalar product between $c(\varphi)$ and the unit normal $n(\varphi) = (\cos(\varphi), \sin(\varphi))$ one has a simple geometric interpretation: $h(\varphi)$ is the distance of the tangent at $c(\varphi)$ from the origin.

## 2. Background And Explanations

A convex body in $\mathbb{R}^n$ is a compact subset $\mathbb{B}$ having non-empty interior and such that it includes the line segment joining any two of its points. A hyperplane $H$ in $\mathbb{R}^n$ is called a supporting hyperplane of $\mathbb{B}$ if it contains a point of $\mathbb{B}$ and if $\mathbb{B}$ is included in one of the two halfspaces defined by $H$. It is not difficult to show that every boundary point of $\mathbb{B}$ lies on at least one supporting hyperplane, and that $\mathbb{B}$ is the intersection of all such halfspaces.

A smooth, closed, planar curves $c$ is called *convex* if its tangent at each point intersects $c$ only at that one point. The complement in $\mathbb{R}^2$ of such a curve has a single bounded component, the *interior* of the curve, and one unbounded component, its *exterior*. The curve is the boundary of its interior, and we denote by $\mathbb{B}$ the curve together with its interior. It is easy to see that $\mathbb{B}$ is a convex body in $\mathbb{R}^2$, as defined above, and in fact the tangent line at any point of $c$ is the unique supporting hyperplane (= line!) containing that point. (There are of course more general planar convex bodies. For example if $P$ is a closed polygon in $\mathbb{R}^2$ together with its interior, then $P$ is a convex body, but there are infinitely many supporting lines through each vertex, while the supporting line containing an edge contains infinitely many points.)

Now let $O$ be some interior point of $c$ and take $O$ as the origin of a cartesian coordinates by fixing a ray from $O$ as the positive $x$-axis. With respect to these coordinates, at each point $p$ on $c$ the outward directed unit normal at $p$ will have the form $n(\varphi) = (\cos(\varphi),\ \sin(\varphi))$ where $\varphi = \varphi(p)$ satisfies $0 \leq \phi \leq 2\pi$. If we as usual think of $\mathbb{S}^1$ as the interval $[0, 2\pi]$ with endpoints identified, then it can be shown that the map $p \mapsto \varphi(p)$ is a smooth one-to-one map of $c$ with $\mathbb{S}^1$, so that the inverse map gives a parametrization $c(\varphi)$ of the curve by $\mathbb{S}^1$. (This just says that given any direction in the plane, there is a unique point $p$ on $c$ where the outward normal has that direction, and the point $p$ varies smoothly with the direction.)

The Minkowski *support function* for the curve $c$ is the function $h$ defined on $\mathbb{S}^1$ by letting $h(\varphi)$ be the distance from the origin of the line of support (or tangent) through $c(\varphi)$, that is $h(\varphi) := n(\varphi) \cdot c(\varphi)$, the scalar product of $c(\varphi)$ and $n(\varphi)$. From this definition it is easy to reconstruct the curve in terms of its support function as in part 1.

## 3. Things To Observe

Recall one has in any parametrization the curvature formula

$$n'(t) = \kappa(t)c'(t),$$

which in the present case reduces to:

$$1/\kappa(\varphi) = h(\varphi) + h''(\varphi) = |c'(\varphi)|.$$

Clearly $aa$ has to be large enough to make $\kappa$ positive and the parametrization nonsingular. Adding a linear combination of $\cos(\varphi)$ and $\sin(\varphi)$ to the support function corresponds to a change of only the origin, the shape of the curve stays the same. The $bb\cos(\varphi)$-term in the support function is therefore not really necessary, but one can use it to see how the parametrization of the curve changes.

The cos-terms in even multiples of $\varphi$ make up the even part $(h(\varphi) + h(\varphi + \pi))/2$ of $h$. The origin is the

*midpoint* of curves with even support function. If $h$ is odd except for the constant term, i.e.,
$$h(\varphi) = aa + (h(\varphi) - h(\varphi + \pi))/2,$$
then one obtains curves of constant width $w$ where:
$$w = h(\varphi) + h(\varphi + \pi) = 2\,aa.$$
The default curve in 3D-XplorMath is such a curve of constant width and the default morph shows a family of such curves. We emphasize the width of our curves by drawing them together with their pairs of parallel tangents. Since the (non-)constancy of the distance between these parallel tangents is difficult to see we have added a circle of the same width (= diameter). One cannot easily recognize how many extrema the curvature $\kappa(\varphi)$ has. To see it clearly we recommend selecting the entry *Show Osculating Circles* from the Action Menu, since the evolute has a cusp at every extremal value of $\kappa$.