Space Curves of Constant Curvature on Cylinders*

These Curves are special cases of the ones described in *Space Curves of Constant Curvature on Tori*, but the situation simplifies so much that they deserve special attention.

First we roll the plane onto a cylinder of radius $R = 1/bb$:

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ R\cos(y/r) \\ R\sin(y/R) \end{pmatrix}.$$

In the plane we describe a curve by its rotation angle against the x-axis, $\alpha(s) = \int_0^s \kappa_g(\sigma)d\sigma$, where κ_g is the curvature of the plane curve, or its geodesic curvature when rolled onto the cylinder:

$$c'(s) := \begin{pmatrix} \cos(\alpha(s)) \\ \sin(\alpha(s)) \end{pmatrix}, \;\; c(s) := \int_0^s c'(\sigma)d\sigma.$$

The cylinder has normal curvature 0 in the x-direction and $1/R$ in the y-direction. The space curvature κ of $F \circ c$ is therefore given by

$$\kappa^2 = \sin^4(\alpha(s))/R^2 + \kappa_g^2(s) = \sin^4(\alpha(s))/R^2 + (\alpha'(s))^2.$$

This is a first order ODE for $\alpha(s)$, if we want $\kappa = const = dd$. The default morph of 3D-XplorMath varies dd.

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

This ODE is harmless, if we look for curves with $\kappa > 1/R$:

$$\alpha'(s) = +\sqrt{\kappa^2 - \sin^4(\alpha(s))/R^2} > 0.$$

The solution curves are, in the plane, convex curves. They reach $\alpha = \pi/2$ in finite time. They are closed because the normals at $\alpha = 0$ and at $\alpha = \pi/2$ are lines of reflectional symmetry.

To discuss curves with $\kappa \leq 1/R$, we differentiate the square of the ODE and cancel $2\alpha'(s)$:

$$\alpha''(s) = -2\sin^3(\alpha(s))\cos(\alpha(s))/R^2.$$

This is a Lipschitz-ODE with unique solutions for any given initial data.
If we choose $\kappa < 1/R$, then the second order ODE forces $\alpha'(s)$ to change sign when $\alpha(s)$ reaches $\alpha_{\max}$ given by $\sin^2(\alpha_{\max}) = \kappa/R < 1$. The solution curves oscillate around a parallel to the x-axis and look a bit like sin-curves.
If we choose $\kappa = 1/R$, then $\alpha_{\max} = \pi/2$. We see that $\alpha(s) := \pi/2$ is a solution of the second order ODE. Hence, any solution which starts with $\alpha(0) < \pi/2$ cannot reach $\pi/2$ in finite time, but converges to $\pi/2$ asymptotically. The corresponding curve $F \circ c$ therefore spirals towards one of the circle-latitudes of the cylinder - unexpectedly?

H.K.