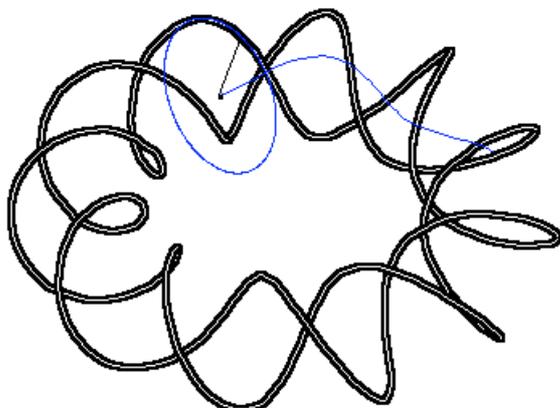


Space Curves of Constant Curvature *



2 - 11 Torus Knot of constant curvature.

See also:

About Spherical Curves

Definition via Differential Equations. Space Curves that 3DXM can exhibit are mostly given in terms of explicit formulas or explicit geometric constructions. The differential geometric treatment of curves starts from such examples and defines geometric properties, i.e., properties which do not change when the curve is transformed by an isometry (= distance preserving map, also called a rigid motion) of Euclidean space \mathbb{R}^3 . The most important such properties are the curvature function κ and the torsion function τ . Once they have been defined one proves the *Fundamental Theorem of Space Curves*, which states that for any given continuous functions κ, τ there is a space curve with these curvature and torsion functions, and, that this curve is uniquely determined up to a rigid motion. In 3DXM we obtain closed constant curvature space curves from:

$$\kappa(t) := aa,$$

$$\tau(t) := bb + cc \cdot \sin(t) + dd \cdot \sin(2t) + ee \cdot \sin(3t).$$

* This file is from the 3D-XplorMath project. Please see:

<http://3D-XplorMath.org/>

We recall the definition of curvature and how the ODE shows up. At each point of a parametrized space curve $c(t)$ there is a parametrized circle $\gamma(t)$ with

$$c(t_0) = \gamma(t_0), \quad \dot{c}(t_0) = \dot{\gamma}(t_0), \quad \ddot{c}(t_0) = \ddot{\gamma}(t_0).$$

This circle – which may degenerate to a straight line – is called the *osculating circle at t_0* , its radius is called *curvature radius at t_0* and the inverse of the radius is called *the curvature at t_0* , $\kappa(t_0)$. The computation of curvature is simpler if the curve is parametrized by arc length, i.e. if the length of all tangent vectors is one, $|\dot{c}(t)| = 1$. One gets $\kappa(t) = |\ddot{c}(t)|$. Check this for the circle (arclength parametrization) $c(t) := r \cdot (\cos(t/r), \sin(t/r))$. The most common way to proceed is to assume that $\kappa(t) > 0$. This allows to define the *Frenet basis* along the curve:

$$\begin{aligned} e_1(t) &:= \dot{c}(t), \\ e_2(t) &:= \ddot{c}(t)/\kappa(t), \\ e_3(t) &:= e_1(t) \times e_2(t). \end{aligned}$$

The Frenet basis defines three curves $t \mapsto e_j(t)$ on the unit sphere. To emphasize the fact that $e_j(t)$ are to be considered as vectors, not as points, one calls the length of their derivative, $|\dot{e}_j(t)|$, *angular velocity* or *rotation speed* and not just velocity. For example, the formula $\ddot{c}(t) = \kappa(t)e_2(t)$ says that $\kappa(t)$ is the rotation speed of $\dot{c}(t)$. We get from $\dot{e}_1(t) \sim e_2(t)$ that the derivative $\dot{e}_3(t) = e_1(t) \times \dot{e}_2(t)$ is proportional to $e_2(t)$. The proportionality factor, the rotation speed of $e_3(t)$, is called the torsion function $\tau(t)$ of the curve $c(t)$. In formulas: $\tau(t) := \langle \dot{e}_3(t), e_2(t) \rangle$.

Having made these definitions, one can change the point

of view and consider the two functions κ, τ as given. This turns the equations that were originally *definitions* of κ and τ into *differential equations* for the curve, the famous

Frenet-Serret Equations:

$$\begin{aligned}\dot{e}_1(t) &= \kappa(t) \cdot e_2(t), \\ \dot{e}_2(t) &= -\kappa(t) \cdot e_1(t) - \tau(t) \cdot e_3(t), \\ \dot{e}_3(t) &= \tau(t) \cdot e_2(t),\end{aligned}$$

or, more compactly with the angular velocity vector

$$\begin{aligned}\vec{\omega}(t) &:= -\tau(t) \cdot e_1(t) + \kappa(t) \cdot e_3(t), \\ \dot{e}_j(t) &= \vec{\omega}(t) \times e_j(t), \quad j = 1, 2, 3.\end{aligned}$$

Finally $\dot{c}(t) := e_1(t)$.

For given continuous functions κ, τ these differential equations have — for given orthonormal initial values — unique orthonormal solutions $\{e_1(t), e_2(t), e_3(t)\}$.

The curve $c(t) := \int^t e_1(s) ds$ is then parametrized by arc length and has the given curvature functions κ, τ .

To find *closed curves of constant curvature*, we employ symmetries. The simplest case is reflection symmetry in normal planes of the curve. When crossing such symmetry planes the torsion, in our case

$$\tau(t) := bb + cc \cdot \sin(t) + dd \cdot \sin(2t) + ee \cdot \sin(3t),$$

changes sign and is skew symmetric w.r.t. those points. Therefore we need $bb = 0$. The symmetry points then are at $t = n \cdot \pi$, $n \in \mathbb{Z}$ and all symmetry planes pass through the intersection line of two neighboring ones. If and only if

the planes intersect in rational angles do the curves close. Therefore we find many closed curves in any 1-parameter family (if $bb = 0$), see e.g. the **default morph**.

With the τ -function that we chose, there is another type of symmetry: If we set $dd = 0$ then τ is even w.r.t. the parameters $t = \pi/2 + n \cdot \pi$, $n \in \mathbb{Z}$ and this implies that 180° rotation around $e_2(t)$ is a symmetry of the curve. Closed curves with this symmetry can be obtained from the Action Menu in the Submenu **Other Closed Curves**. If one chooses in the Action Menu **Add Symmetry Elements** then these normal rotation axes are added to the picture. The closing condition is easy to see:

- (i) The symmetry normals have all to be in the same plane and therefore all will intersect in one point.
- (ii) The intersection angle between two adjacent normals has to be rational.

For curves with this normal symmetry the parameter bb , the mean value of the torsion τ , has a very nice property. To see it, select one of the curves with normal symmetry, then **Add Symmetry Elements**. Next go to **Set Morphing**, the 2nd entry of the Settings Menu, click the button **Init to Current Parameters** and finally increase the upper morphing bound $b1$ by about 0.2. Preferably choose one of the stereo options and start morphing: The symmetry normals will move apart but remain orthogonal to a common, helicoidal, symmetry axis.

This regular behavior allowed us to program the morph **bb Keeps SymLines Colinear**, the last entry in the Ani-

mate Menu. Start with a closed curve with normal symmetries (from the submenu of the Action Menu). In **Set Morphing** click again the button **Init to Current Parameters**, but this time increase the upper bound $c1$ by about 0.2. Click **OK** and click the last entry in the **Animate Menu**. *In the resulting deformation the program adjusts the mean value bb of the torsion in such a way that the symmetry normals remain collinear!!*

This feature allows the user to find other closed curves: Watch the morph until the symmetry normals appear to form again a rational angle, stop the morph by clicking the mouse. The parameters at which you stopped are now the current parameters. Click **Init to Current Parameters** in **Set Morphing** and now increase or decrease $c1$ by about 0.02, do **OK** and run the last-entry-morph again. With few such steps the new curve can be visibly closed. One last time **Init to Current Parameters** and now do in the **Animate Menu** **aa, bb Keep Curves closed**. This entry is there to deform closed curves as closed curves, but in the current use it will improve the visibly closed curve to being high precision closed. – Some of the closed examples which the submenu offers, have been obtained with this method. The last example, the 11-2-knot, needed such a large value of the mean value bb of $\tau(t)$ that the torsion of this example has no sign changes! It is easier to imagine examples like **6 helices**, which look more as one would expect: being made up of left winding and right winding pieces of helices, joined by pieces of circles.

See the tutorial movie on constant curvature space curves at <http://3d-xplormath.org/Movies/index.html> Do not miss to select **Show Osculating Circles & Evolute**. The constant radius of the osculating circles shows the constant curvature and the rotating motion of the radius shows size and sign of the torsion.

In 3DXM one can choose in the Action Menu **Parallel Frame**. This frame is designed to rotate as little as possible along the curve, in \mathbb{R}^3 . This property is more obvious when one looks at the torus knots than at the constant curvature curves. For further details see curves of *constant torsion*. The main advantage of these parallel frames is that they neither make it necessary to assume more than continuity of the second derivative \ddot{c} , nor that $\kappa > 0$ everywhere, even straight lines are not exceptional curves if one works with these frames. Their differential equation is also simple:

Frenet-Serret Equations for Parallel Frames:

$$\dot{e}_1(t) := a(t) \cdot e_2(t) + b(t) \cdot e_3(t),$$

$$\dot{e}_2(t) := -a(t) \cdot e_1(t),$$

$$\dot{e}_3(t) := -b(t) \cdot e_1(t).$$

With an antiderivative $T(t)$ of the torsion $\tau(t) = T'(t)$ we can of course write the twodimensional curvature vector $(a(t), b(t))$ in terms of $\kappa(t), \tau(t)$, namely:

$$(a(t), b(t)) := \kappa(t) \cdot (\cos(T(t)), \sin(T(t))).$$

H.K.