Clifford Tori *

a) Parametrized by Curvature Lines, b) Hopf-fibered

Clifford Tori are embeddings of the torus into the unit sphere $\mathbb{S}^3$ of $\mathbb{R}^4$, by $(u,v) \to F(u,v) := (w,x,y,z)$, where

$$F_{\text{Clifford}}(u,v) = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}(u,v) = \begin{pmatrix} \cos\alpha\cos u \\ \cos\alpha\sin u \\ \sin\alpha\cos v \\ \sin\alpha\sin v \end{pmatrix}$$

$\alpha :=\ aa + bb\sin(ee\cdot 2\,v),\ bb \neq 0$ for Bianchi-Pinkall Tori.

(Note that this is, for $bb = 0$, the product of a circle in the (w,x) plane with a second circle in the (y,z) plane, and so is clearly flat.) To get something that we can see in $\mathbb{R}^3$, we stereographically project $\mathbb{S}^3 \mapsto \mathbb{R}^3$; i.e., the Clifford tori in $\mathbb{R}^3$ are the embeddings $(u,v) \to P(F(u,v))$, where $P:\mathbb{S}^3 \to \mathbb{R}^3$ is Stereographic projection.
We take as the center of the stereographic projection map the point $(\cos(cc\,\pi), 0, \sin(cc\,\pi), 0)$. Varying cc deforms a torus of revolution through cyclides. The `Default Morph` varies aa, hence changes the ratio of the two circles.
Another morph, `Conformal Inside-Out Morph` (also in the Animation Menu), is in $\mathbb{R}^4$ a rotation (parametrized by $0 \leq ff \leq 2\pi$), that moves the torus through the center of the stereographic projection. The image in $\mathbb{R}^3$ therefore passes through infinity: we see a torus with one puncture that has a flat end. It looks like a plane with a handle.

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* This file is from the 3D-XplorMath project. Please see:
http://3D-XplorMath.org/

The Clifford tori (in $\mathbb{S}^3$) are fibered by *Great Circles*, the Hopf fibers, $u + v = const.$ These Great Circles are of course the *asymptote lines* on the tori. We show two versions of the stereographically projected Clifford tori: a) parameterized by curvature lines and b) by Hopf fibers. (To get the explicit parametrization of the latter, take $F(u + v, u - v)$ in the above formulae.)

The classical Clifford Torus corresponds to $\alpha = aa = \pi/4$. It has maximal area among the family and divides $\mathbb{S}^3$ into two congruent solid tori. But the other torus-leaves of the foliation, obtained by varying aa, are also interesting. All of them are foliated by *Clifford-parallel* great circles and hence flat. They are special cases of the flat Bianchi-Pinkall Tori in $\mathbb{S}^3$ (visible after stereographic projection) and discussed in more detail in their ATO ("About This Object..."), see the Documentation Menu.

Why is the ff-morph a `Conformal Inside-Out Morph`? A compact surface divides $\mathbb{R}^3$ in two components and the bounded component is called the inside. One surface of the family, the once punctured torus that passes through infinity, divides $\mathbb{R}^3$ into two congruent unbounded components. This surface has no inside and at this moment in the deformation inside and outside are interchanged. The 180 degree rotation in the rotation family is, on the $\pi/4$-torus, a conformal anti-involution. It has a Hopf fiber as connected fixed point set. Use in the Action Menu `Surface Coloration` and choose the default `two-sided user coloration` which emphasizes the fixed fiber.

H.K.

TOC