Cassinian Ovals*

Level function in 3DXM:

$$f(x,y) := (x-aa)^2 + y^2) \cdot ((x+aa)^2 + y^2) - bb^4$$

The default *Color Morph* varies $bb = ff^{1/4}$ instead of ff.

The Cassinian Ovals (or Ovals of Cassini) were first studied in 1680 by Giovanni Domenico Cassini (1625–1712, aka Jean-Dominique Cassini) as a model for the orbit of the Sun around the Earth.

A Cassinian Oval is a plane curve that is the locus of all points P such that the *product of the distances* of P from two fixed points F_1, F_2 has some constant value c, or $\overline{PF_1}\,\overline{PF_2} = c$. Note the analogy with the definition of an ellipse (where product is replaced by sum). As with the ellipse, the two points F_1 and F_2 are called *foci* of the oval. If the origin of our coordinates is the midpoint of the two foci and the x-axis the line joining them, then the foci will have the coordinates $(a, 0)$ and $(-a, 0)$. Following convention, $b := \sqrt{c}$. Then the condition for a point $P = (x, y)$ to lie on the oval becomes: $((x-a)^2+y^2)^{1/2}((x+a)^2+y^2)^{1/2} = b^2$. Squaring both sides gives the following *quartic polynomial equation* for the Cassinian Oval:

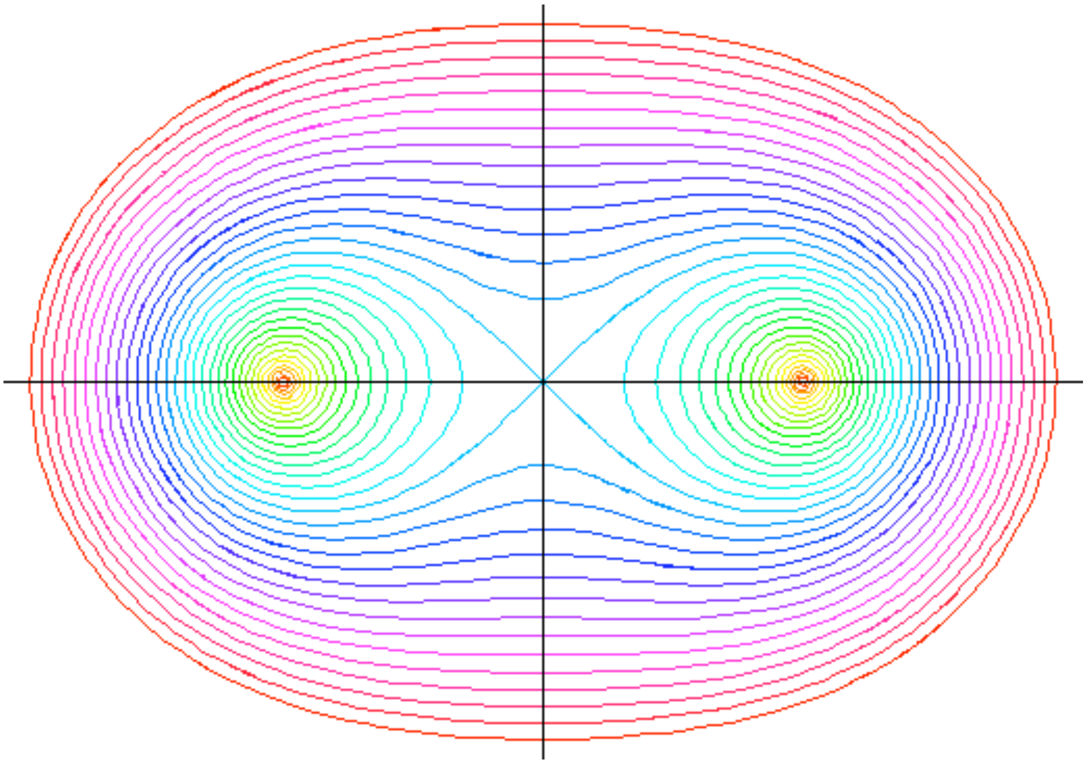
$$((x-a)^2 + y^2)((x+a)^2 + y^2) = b^4.$$

When b is less that half the distance $2a$ between the foci, i.e., $b/a < 1$, there are two branches of the curve. When

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

$a = b$, the curve has the shape of a figure eight and is known as the *Lemniscate of Bernoulli.*

The following image shows a family of Cassinian Ovals with $a = 1$ and several different values of b.



In 3D-XplorMath, you can change the value of parameter $b = bb$ in the Settings Menu $\rightarrow$ SetParameters. An animation of varying values of b can be seen from the Animate Menu $\rightarrow$ Color Morph.

Bipolar equation: $r_1 r_2 = b^2$

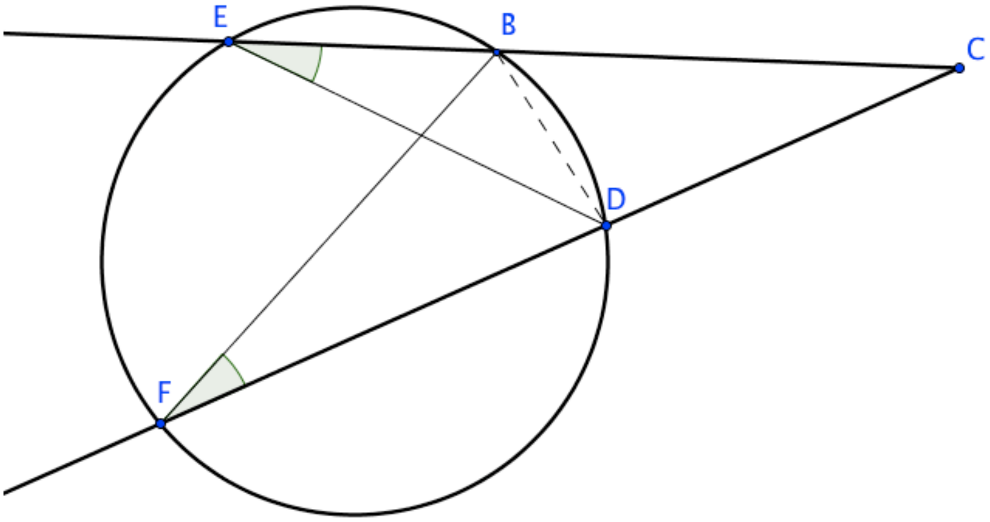
Polar equation: $r^4 + a^4 - 2r^2 a^2 \cos(2\theta) = b^4$

A parametrization for Cassini's oval is $r(t) \cdot (\cos(t), \sin(t))$,

$$r^2(t) := a^2 \cos(2t) + \sqrt{(-a^4 + b^4) + a^4 (\cos(2t))^2},$$

$t \in (0, 2\pi]$, and $a < b$. This parametrization only generates parts of the curve when $a > b$.

By default 3D-XplorMath shows how the product definition of the Cassinian ovals leads to a *ruler and circle* construction based on the following circle theorem about products of segments:



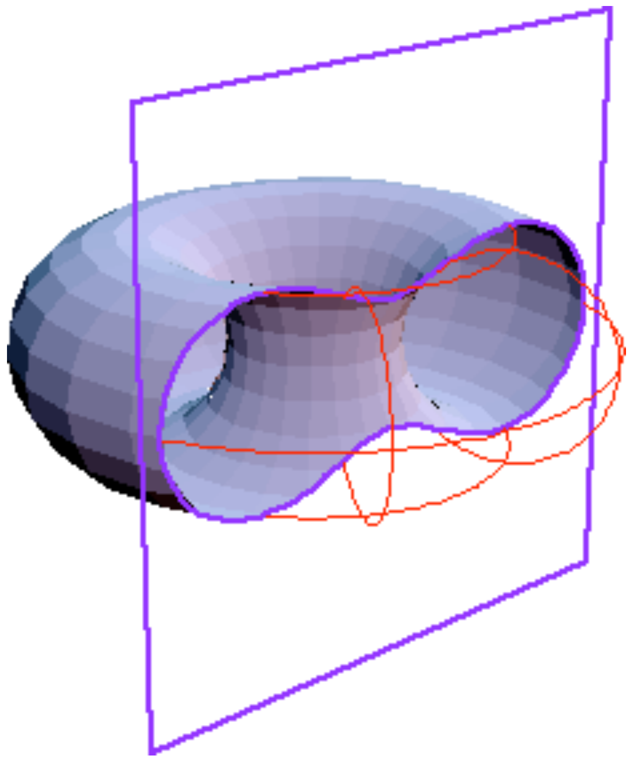
Cassinian Ovals as sections of a Torus

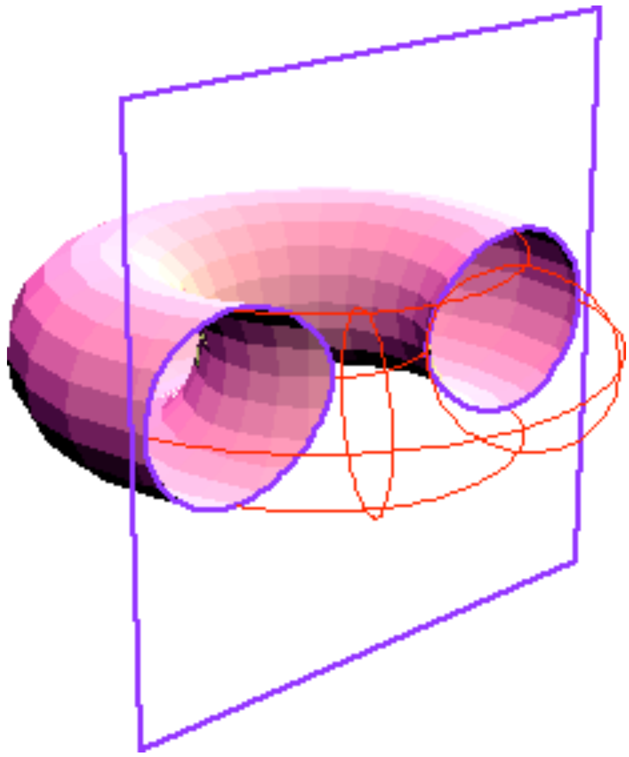
Let c be the radius of the generating circle and d the distance from the center of the tube to the directrix of the torus. The intersection of a plane c distant from the torus' directrix is a Cassinian oval, with $a = d$ and $b^2 = \sqrt{4}cd$, where a is half of the distance between foci, and b^2 is the constant product of distances.

Cassinian ovals with a large value of b^2 approch a circle, and the corresponding torus is one such that the tube radius is larger than the center to directrix, that is, a self-intersecting torus without the hole. This surface also approaches a sphere.

Note that the two tori in the figure below are not identical.

Arbitrary vertical slices of a torus are called Spiric Sections. In general they are *not* Cassinian ovals.





Proof: Start with the equation of a torus

$$(\sqrt{x^2+y^2}-d)^2+z^2=c^2.$$

Insert $y=c$, rearrange and square again:

$$x^2+z^2+d^2=2d\sqrt{x^2+c^2},\ (x^2+z^2+d^2)^2=4d^2(x^2+c^2).$$

Now multiply the factors of the implicit equation of an Cassinian oval and rearrange

$$\begin{aligned}((x-a)^2+y^2)\cdot((x+a)^2+y^2)&=b^4,\\(x^2-a^2)^2+y^4+2y^2(x^2+a^2)&=b^4,\\(x^2+y^2)^2+2a^2(y^2-x^2)&=b^4-a^4.\end{aligned}$$

These two equations match because of $a=d,\ b^2=2dc$, after rotation of the y-axis into the z-axis.

Curves that are the locus of points the product of whose distances from n points is constant are discussed on pages 60–63 of Visual Complex Analysis by Tristan Needham.

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