Boy's Surface (Bryant-Kusner)

See Möbius Strip first. The non-orientable surfaces are *one-sided* and this property can best be understood, if one starts from a Möbius Strip.

The Klein Bottle is easier to visualize than the Boy's surface. Each meridian of the Boy's surface is the center line of a narrow Möbius band. To see this, select "Set u,v Ranges" from the Settings menu and for example change umin to -0.998, and vmin to 6.1.

The "equator" of the Boy's surface is a different kind of Möbius band, it has *three* half-twists instead of one. The standard morph begins with this Möbius band and widens it until the Boy's surface is complete:

$$aa = 0.5, \text{ vmin} = 0, \text{ vmax} = 2\pi, \text{ umax} = 1, \text{ and}$$
$$0.9 \geq \text{umin} \geq 0.002$$

"Boy's surface" is really a family of surfaces. Boy, in his dissertation under Hilbert, constructed this surface as the first known *immersion* of the projective plane. Being nonorientable implies that no embedding is possible. Boy's surface has, besides its self-intersection curves, only one more singularity, namely a triple point. Boy's construction was topological.

Apéry has found algebraically embedded "Boy's surfaces". These carry one-parameter families of ellipses.

The Bryant-Kusner Boy's surfaces are obtained by an inversion from a minimal surface in $\mathbb{R}^3$. The minimal surface is an immersion of $\mathbb{S}^2 - \{6 \text{ points}\}$ such that antipodal points have the same image in $\mathbb{R}^3$. The six punctures are three antipodal pairs, and the minimal surface has so called *planar ends* at these punctures. In this context it is important that the inversion of a planar end has the puncture that can be closed *smoothly* by adding one point. The closing of the three pairs of antipodal ends thus gives a triple point on the surface obtained by inversion. Explicitly let

$M(z) = \Re(a(z)V(z)) + (0,0,1/2)$, where

$a(z) = \left(z^3 - z^{-3} + \sqrt{5}\right)^{-1}$ and

$V(z) = \left(i(z^2 + z^{-2}), z^2 + z^{-2}, \frac{2i}{3}(z^3 + z^{-3})\right)$.

Then $Boys(z)$ is obtained by inverting $M(z)$ in the unit sphere:

$$Boys(z) := \frac{M(z)}{||M(z)||^2}.$$

H.K.