Boy Surfaces,

following **Apery** and **Bryant-Kusner** *

See Möbius Strip and Cross-Cap first.

All the early images of the projective plane in $\mathbb{R}^3$ had singularities, the **Boy Surface** was the first *immersion*. Since the projective plane is *non-orientable*, no embedding into $\mathbb{R}^3$ exists and *self-intersection curves* have to occur on the image. In fact, the self-intersection curve of the Boy surface is also *not* embedded, the surface has a *triple point.* Boy discovered the surface while working for his PhD under Hilbert. Boy's construction was differential topology work, his surface has no special local geometry.

Apery found an **Algebraic Boy Surface**. Moreover, his surface is covered by a 1-parameter family of ellipses. This is his *Parametrization*:

$$F_{Apery}(u,v) = \begin{pmatrix} \frac{\cos^2(u)\cos(2v)+\sin(2u)\cos(v)/\sqrt{2}}{\sqrt{2}-\sin(2u)\sin(3v)} \\ \frac{\cos^2(u)\sin(2v)-\sin(2u)\sin(v)/\sqrt{2}}{\sqrt{2}-\sin(2u)\sin(3v)} \\ \frac{\sqrt{2}\cos^2(u)}{(\sqrt{2}-\sin(2u)\sin(3v))} \end{pmatrix}.$$

This parametrization is obtained by restricting the

---

* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

following *even* map from $\mathbb{R}^3$ to $\mathbb{R}^3$ to the unit sphere:

$$denom := (\sqrt{2} - 6xz)(x^2 + y^2) + 8x^3 z$$

$$F_x := ((x^2 - y^2)z^2 + \sqrt{2}xz(x^2 + y^2))/denom$$

$$F_y := (2xyz^2 - \sqrt{2}yz(x^2 + y^2))/denom$$

$$F_z := z^2(x^2 + y^2)/denom$$

The image of the unit sphere is also an *image of the projective plane* since

$$(F_x, F_y, F_z)(-p) = (F_x, F_y, F_z)(p).$$

The `Default Morph` starts with a band around the equator, which is a Möbius Strip with *three* halftwists. The complete surface is obtained by attaching a disk (centered at the polar center). 3DXM supplies a second morph, `Range Morph` in the Animation Menu. It starts with a band around a meridian, which is another Möbius Strip with *one* halftwist. This Möbius Strip is moved over all the meridians, covering the surface with embedded Möbius Strips.

Bryant-Kusner Boy Surfaces are obtained by an inversion from a minimal surface in $\mathbb{R}^3$. The minimal surface is an immersion of $\mathbb{S}^2 - \{6 \text{ points}\}$ such that antipodal points have the same image in $\mathbb{R}^3$, so that the minimal surface is an image of the projective plane minus three points. The six punctures are three antipodal pairs, and the minimal surface has so called *planar ends* at these punctures. This is the same as saying that the Weierstrass-integrand has no residues, hence can be explicitly integrated. In this context it is important that the inversion of a planar end has a puncture that can be *smoothly* closed by adding one point. The closing of the three pairs of antipodal ends thus gives a triple point on the smoothly immersed surface which is obtained by inverting the minimal surface.

As `Default Morph` and as `Range Morph` we took the same deformations as in the algebraic case. The first emphasizes the equator Möbius Strip with *three* half-twists, the second covers the surface with embedded Möbius Strips that have meridians as center lines.

A *Parametrization* is obtained by first describing the minimal surface as an image of the Gaussian plane, then invert it in the unit sphere. Parameter lines come by taking polar coordinates in the unit disk.

$MinSurf(z) := \mathrm{Re}\,(V(z)/a(z)) + (0,0,1/2)$, where
$a(z) := \bigl(z^3 - z^{-3} + \sqrt{5}\bigr)$ and
$V(z) := \bigl(i(z^2 + z^{-2}),\ z^2 + z^{-2},\ \frac{2i}{3}(z^3 + z^{-3})\bigr)$.
Finally the inversion:

$$Boy(z) := \frac{MinSurf(z)}{||MinSurf(z)||^2}.$$

Finally we describe an ODE which allows to compute the self-intersection curve as long as the two normals along the self-intersection are not parallel – in other words, as long as the self-intersection is transversal.

Let $F : D^2 \mapsto \mathbb{R}^3$ be a parametrized surface with unit normal field $N : D^2 \mapsto \mathbb{R}^3$. Let $p, q \in D^2$ be any two points with $N(p) \neq \pm N(q)$.
The vector $T := N(p) \times N(q)$ is then tangent to the surface at $F(p)$ and $F(q)$.
Therefore we have unique vectors $\dot{p} \in T_p D^2, \dot{q} \in T_q D^2$ such that $T = dF_p \cdot \dot{p} = dF_q \cdot \dot{q}$.
This defines an ODE on $D^2 \times D^2$ with singularities where $N(p) = \pm N(q)$.

Claim: Along the pair of solution curves we have
$$F(p(t)) - F(q(t)) = const.$$

Clearly:
$(F(p(t)) - F(q(t)))^{\cdot} = dF_p \cdot \dot{p} - dF_q \cdot \dot{q} = T - T = 0.$

In particular, if we start at a self-intersection point $F(p(0)) = F(q(0))$, then the solution curve runs along the self-intersection, $F(p(t)) = F(q(t))$.
In case one does not have an explicit self-intersection point one can start from nearby points, meaning the distance between $F(p)$ and $F(q)$ is small. On the intersection of the tangent planes at these points find

the point C which is closest to the segment $\overline{F(p)F(q)}$. Use $C - F(p) = dF_p \cdot \Delta p$ and $C - F(q) = dF_q \cdot \Delta q$ to get points $p + \Delta p, q + \Delta q$ which are much closer to a self-intersection point. Iteration of this procedure converges rapidly.

H.K.