The Spherical Helicoids

A general helicoid can be obtained by applying a screw motion to a planar curve. Suppose the curve is given as $s \mapsto (f(s), g(s))$, then the helicoid can be written as

$$F(s,t) = (f(s)\cos t, f(s)\sin t, g(s) + ht)$$

with a non-zero constant h. The surface which is generally called *the* helicoid arises as the special case $f(s) = s$ and $g(s) = 0$. Our more general class of helicoids is useful because it allows to construct concrete examples of surfaces that are otherwise elusive.

For example, let us specialize a little by taking $f(s) = s$ (so that the curve is a graph over the t-axis). Then the Gauss curvature of these helicoids is

$$K = \frac{-h^2 + s^3\, g'(s)\, g''(s)}{\left(h^2 + s^2 + s^2\, g'(s)^2\right)^2}$$

This is just a first order ODE for $g'(s)$, and it is easy to check that

$$g'(s) = \sqrt{-1 - \frac{h^2}{s^2} + \frac{1}{a - \sqrt{K}\, s^2}}$$

is a 1-parameter family of solutions in the case that K is a (positive) constant. The family parameter is a. Observe that the additional integration constant we get by integrating g' only amounts to a vertical translation of the surface.

As all *complete* surfaces of positive constant curvatures are round spheres, any other example must necessarily develop singularities. This was for a long time a perfectly good reason to ignore them, and now it has been for a while a perfectly good reason to find them interesting.

For our spherical helicoids, the singularity arises as the curve $s = s_0$ for the value of s_0 where the integrand $g'(s)$ becomes 0, and is thus a horizontal circle.

The surface appears as an exercise in Eisenhart's *A Treatise On The Differential Geometry of Curves And Surfaces*. It was certainly well known much earlier.

M.W.