

Pseudosphere

The Pseudosphere is a surface of revolution of Gaussian curvature -1, or in other words, the product of its principal curvatures is -1. On a surface of revolution this translates into a simple analytic property: Parametrize the meridian curve by arclength $s \mapsto (r(s), h(s))$, $r'^2 + h'^2 = 1$. Then r is a solution of the differential equation $r'' = r$. Consequently h is also known, $h = \int^s \sqrt{1 - r'(u)^2} du$.

The Pseudosphere is best known because its intrinsic geometry is hyperbolic, the meridians are a family of asymptotic geodesics and the orthogonal latitudes are therefor a geodesically parallel family of “horocycles”, i.e. limits of circles as their midpoints converge to the limit point of the asymptotic geodesics.

This Pseudosphere can also be obtained by the construction that relates solutions of the Sine-Gordon equation to surfaces of Gaussian curvature -1. For the Pseudosphere, the appropriate Sine-Gordon solution is the so-called 1-soliton (or travelling-wave) solution:

$$q(u, v) := 4 \cdot \arctan(\exp(u)).$$

The obtained parametrization has another remarkable property: The diagonal curves in **all** the parameter quadrilaterals have the same length! Nets used for fishing also have such equi-quadrilaterals as meshes; the mathematical term is “Tchebycheff net”. Such Tchebycheff nets exist on all surfaces which are isometric immersions of (portions of) the hyperbolic plane. This fact plays a key role in the proof of Hilbert’s theorem, which says: There is no smooth isometric immersion of the whole hyperbolic plane into euclidean 3-space.

See also the “Tractrix” under *Plane Curves*.

H.K.