Hopf Fibration and Clifford Translation of the 3-sphere

Most rotations of the 3-dimensional sphere $\mathbb{S}^3$ are quite different from what we might expect from familiarity with 2-sphere rotations. To begin with, most of them have no fixed points, and in fact, certain 1-parameter subgroups of rotations of $\mathbb{S}^3$ resemble *translations* so much, that they are referred to as *Clifford translations*. The description by formulas looks nicer in complex notation. For this we as usual identify $\mathbb{R}^2$ with $\mathbb{C}$, with multiplication by $\mathbf{i}$ in $\mathbb{C}$ represented in $\mathbb{R}^2$ by matrix multiplication by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Then the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ is given by:

$$\mathbb{S}^3 := \{p = (z_1, z_2) \in \mathbb{C}^2;\ |z_1|^2 + |z_2|^2 = 1\}$$
$$\sim \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4;\ \sum (x_k)^2 = 1\}.$$

And for $\varphi \in \mathbb{R}$ we define the Clifford Translation $C_\varphi : \mathbb{S}^3 \to \mathbb{S}^3$ by $C_\varphi(z_1, z_2) := (e^{\mathbf{i}\varphi} z_1,\ e^{\mathbf{i}\varphi} z_2)$.

The orbits of the one-parameter group C_φ are all great circles, and they are equidistant from each other in analogy to a family of parallel lines; it is because of this behaviour that the C_φ are called Clifford translations.

But in another respect the behaviour of the C_φ is quite different from a translation—so different that it is difficult to imagine in $\mathbb{R}^3$. At each point $p \in \mathbb{S}^3$ we have one 2-dimensional subspace of the tangent space of $\mathbb{S}^3$ which is orthogonal to the great circle orbit through p. A Euclidean translation would simply translate these normal spaces into each other, but a Clifford translation rotates them so that the velocity of the translation along the orbit is equal to the angular velocity of the rotation of the normal spaces. This normal rotation is responsible for a very curious fact: *Any two orbits are linked.*

The fact that any two orbits are equidistant permits us to make the set of orbits into a metric space., and one can check that this space is isometric to the sphere of radius one-half in $\mathbb{R}^3$. Therefore one can map $h : \mathbb{S}^3 \to \mathbb{S}^2$ by mapping $p \in \mathbb{S}^3$ to its orbit, identified as a point of $\mathbb{S}^2$, and one can write this mapping in coordinates as:

$$h(z_1, z_2) = (|z_1|^2 - |z_2|^2, \mathrm{Re}\ (z_1 z_2), \mathrm{Im}\ (z_1 z_2))$$

This map h is called the *Hopf map* (or Hopf Fibration), and the orbits, the fibres of this map, are called *Hopf fibres*. It is named for Heinz Hopf, who studied it in detail, and found the completely unexpected fact that this map could not be deformed to a constant map.

The visualization in 3D-XplorMath shows four tori each of which is made up of Hopf fibres. We emphasize this with the coloration: each fibre has a constant colour and the colour varies with the distance of the fibres. One can see that any two of the four tori are linked, and one can also see that any two fibres on any one such torus are linked. Since 3D-XplorMath visualizes objects in $\mathbb{R}^3$ and not $\mathbb{S}^3$, before rendering the tori we first map them into $\mathbb{R}^3$ using the stereographic projection mapping $\mathbb{S}^3 \to \mathbb{R}^3$:

$$(x_1, x_2, x_3, x_4) \to (x_1, x_2, x_3)/(1 + x_4).$$

H.K.