Bianchi-Pinkall Flat Tori in $\mathbb{S}^3$

1. Parameter Dependent Formulas in 3DXM

We can parametrize $\mathbb{S}^3$, considered as a submanifold of $\mathbb{C}^2$, by:

$$F(u, \alpha, v) = (\cos(\alpha)e^{iu}e^{iv}, \sin(\alpha)e^{iu}e^{-iv}),$$

where $u \in [0, 2\pi)$, $\alpha \in [0, \pi/2]$, and $v \in [0, \pi]$. We will get the Pinkall Tori first as flat tori in $\mathbb{S}^3$ by taking α to be a function of v, $\alpha := aa + bb \sin(\mathrm{ee}\, 2v)$ (although the theory allows more general choices.) Next we stereographically project $\mathbb{S}^3$ from

$$p = (\cos(cc \cdot \pi), 0, \sin(cc \cdot \pi), 0)$$

to get **conformal** images of the flat tori in $\mathbb{S}^3$. The lines $v = \mathit{const}$ are circles, the stereographic images of the Hopf circles $u \mapsto F(u, \alpha, v)$.

Finally, by morphing $0 \leq ff \leq 2\pi$, we can isometrically rotate $\mathbb{S}^3$ so that the Hopf circle $v = 0$ is the rotation axis. The stereographic image of this rotation is a conformal transformation of $\mathbb{R}^3 \cup \{\infty\}$ which "rotates" $\mathbb{R}^3$ around a circle on the pictured torus. In the case $aa = \pi/4$ we obtain for $ff = 0$

and $ff = \pi$ the same torus, but inside and outside interchanged. This is best viewed with the default 'Two Sided User Coloration'. It can be selected from a submenu of the Action Menu.

2. Background and Explanations

The tori that we usually see are, from the point of view of complex analysis, rectangular tori, meaning that they have an orientation reversing symmetry and the set of fixed points of this symmetry has two components. (The better known tori of revolution have isometric reflections with **two** circles as fixed point sets.) Of course one tries to deform these tori to obtain non-rectangular ones. Obviously one can destroy the mirror symmetry, but this does not imply that one gets tori with a non-rectangular complex structure. The first proof, by Garcia, that one can embed all tori in $\mathbb{R}^3$ was non-constructive and difficult.

A simpler and constructive way to get tori with arbitrary conformal type was found by Pinkall, whose idea was to construct tori that are flat in $\mathbb{S}^3$ (and hence have an easy way to compute their conformal type from their flat geometry), and then stereographically project them to $\mathbb{R}^3$. While the resulting tori are no longer flat, this does preserve their conformal type.

The construction of flat surfaces in $\mathbb{S}^3$ goes back to 1894, when Bianchi classified all flat immersions in $\mathbb{S}^3$. In particular, he realized that the two families of asymptotic lines of a flat surface in $\mathbb{S}^3$ are left translations of a pair of curves that are either great circles or have constant torsion $+1$ and -1, respectively. The left translations arise by viewing $\mathbb{S}^3$ as the group of unit quaternions. An open problem for Bianchi was to determine when his flat surfaces were closed.

The first case when one of the curves is a great circle is of special interest for this problem. To explain why, we will need the Hopf fibration. Thinking of $\mathbb{S}^3$ as being part of $\mathbb{C}^2$, we can multiply points of $\mathbb{S}^3$ by e^{iu}, thus fibering $\mathbb{S}^3$ with circles, the Hopf circles, and the set of all such circles forms a metric space with distance being the distance between the Hopf circles in $\mathbb{S}^3$. As such it is isometric to a 2-dimensional sphere of radius $1/2$. We thus obtain a natural projection $\mathbb{S}^3 \to \mathbb{S}^2$, the Hopf map. It can be written as $(z_1, z_2) \mapsto z_1/z_2$, where we interpret the range as the Riemann sphere $\hat{\mathbb{C}}$. Moreover, Hopf circles are mapped to Hopf circles by left translations.

Now suppose we have a flat surface in $\mathbb{S}^3$ where one of the generating curves is a great circle. We can arrange $\mathbb{S}^3$ so that this great circle is part of the Hopf fibration, and thus all curves of the same family of asymptotic lines are Hopf circles. The surface in $\mathbb{S}^3$ is thus invariant under the Hopf action and projects to a curve in $\mathbb{S}^2$ under the Hopf map. Vice versa, the preimage of a curve in $\mathbb{S}^2$ under the Hopf map yields a flat surface in $\mathbb{S}^3$. In case the curve in $\mathbb{S}^2$ is closed, the surface in $\mathbb{S}^3$ is a flat torus. (The explanation so far is described in more detail in Spivak IV, p. 139ff.)

Pinkall found a simple way to determine the conformal type of the flat torus in terms of the geometry of the curve in $\mathbb{S}^2$ — in particular it was then easy to see that **all** possible conformal types can occur.

3. Visualizing Parts of the Theoretic Description

We cannot visualize $\mathbb{S}^3$ in such a way that all distances are preserved. We will use stereographic projection from $p = (\cos(cc \cdot \pi), 0, \sin(cc \cdot \pi), 0)$ to map $\mathbb{S}^3 - \{p\}$ one-to-one onto $\mathbb{R}^3$. Recall that: angles are not changed by stereographic projection, circles are mapped to circles or straight lines, and the images

of great circles meet the equator sphere in antipodal points, so many properties of $\mathbb{S}^3$ get represented in geometrically comprehensible ways.
Our parametrization F of $\mathbb{S}^3$ emphasizes the Hopf fibration since the great circles $u \mapsto F(u, \alpha, v)$ are indeed the orbits of the Hopf-action of $\mathbb{S}^1$ on $\mathbb{S}^3$, given by $(u, p) \mapsto e^{iu}p$. Each such "Hopf Fiber" lies in one of the parallel tori $\alpha =$ constant, and the great circles $\alpha \mapsto F(u, \alpha, v)$, meet these ($\alpha =$ constant)-tori orthogonally, so that α measures the distance between them.
We get all the Hopf circles on each α-torus for $0 \leq v \leq \pi$, except that those tori degenerate to just one Hopf circle if $\alpha = 0$ or $\alpha = \pi$. This makes it plausible that $(\alpha, 2v)$ are polar coordinates on the metric space of Hopf circles, on the image $\mathbb{S}^2$ of the Hopf map.
Pinkall observed that closed curves on this image sphere, in polar coordinates given as: $(\alpha(s), 2v(s))$, (with $\alpha(s)$ never equal to 0 or $\pi/2$) allow one to write down immersed tori in $\mathbb{S}^3$ as:

$$(u, s) \mapsto (F(u, \alpha(s), v(s)).$$

For example taking $\alpha(s) = \pi/4$ gives the "Clifford Torus" in $\mathbb{S}^3$, a minimal embedding of the square torus. For other constant $\alpha(s)$ in $(0, \pi/2)$ one gets the above parallel family of α-tori, the lengths of their two orthogonal generators are $2\pi\cos(\alpha)$ and $2\pi\sin(\alpha)$.

On all of these tori we still have that the parameter lines $s =$ constant are Hopf-Fibers, and since these are equidistant (as orbits of an isometric action) it follows that the metric is flat. Pinkall proved that length and area of the curve in $\mathbb{S}^2$ determine the conformal structure of the torus in $\mathbb{S}^3$, hence in $\mathbb{R}^3$, and that all conformal structures occur.

Observe that the usual tori of revolution in $\mathbb{R}^3$ are all rectangular, and most of the Pinkall tori shown by 3D-XplorMath are very different from these. The tori with aa $= \pi/4$ are all rhombic, because they can be rotated into themselves by 180° rotations (in $\mathbb{S}^3$, not in $\mathbb{R}^3$) around any of the Hopf-Fibers on them. A cyclic morph with $0 \le ff \le 2\pi$ rotates around the circle $v = 0$ (we see of course the stereographic image of that rotation). For $ff = \pi$ we get an anti-involution of the torus with the circle as the (connected) fixed point set—only rhombic tori have such anti-involutions. (The square torus is rectangular and rhombic.) In the rhombic case $aa = \pi/4$ we get for $ff = \pi/2$ and $ff = 3\pi/2$ surfaces in $\mathbb{S}^3$ that pass through p so that the stereographic images in $\mathbb{R}^3$ pass through ∞ — otherwise we could not turn the torus inside out continuously.

The program takes $\alpha(v) := aa + bb \sin(\mathrm{ee}\, 2v)$ (with $ee = 3$ for the default image and $ee = 5$ for the default morph), allowing rather different examples.
Again, these tori are shown in $\mathbb{R}^3$ by using the (conformal) stereographic projection of $\mathbb{S}^3 \setminus \{p\} \to \mathbb{R}^3$, where $p = (\cos(cc \cdot \pi), 0, \sin(cc \cdot \pi), 0)$. Morphing cc therefore gives other images of $\mathbb{S}^3$, in particular other conformal images of these tori.

H.K.