About Quadratic Surfaces

Quadratic surfaces in $\mathbb{R}^3$ are the solution sets of quadratic equations

$$h(x,y,z) := Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

Explicit parametrizations are given at the end.
There are poor examples, i.e.
with no solutions: $x^2 + y^2 + z^2 + 1 = 0$,
solutions consisting of a point: $x^2 + y^2 + z^2 = 0$,
or solution sets consisting of a line: $x^2 + y^2 = 0$,
or solution sets like that of a linear function: $x^2 = 0$.
But under mild assumptions, namely that the derivative of h does not vanish on (most of) the solution set, we get more interesting surfaces (possibly with singularities) as solution sets.
For products of linear functions we have intersections of (or parallel) planes: $(x - y + a)(x \pm y + b) = 0$, we may have cylinders over quadratic curves, e.g. if the equation does not contain z: elliptic cylinder $x^2 + y^2 - 1 = 0$, hyperbolic cylinder $x^2 - y^2 - 1 = 0$, parabolic cylinder $x^2 - y = 0$.

There are various cones, namely solution sets which contain for each solution $(x, y, z) \neq (0, 0, 0)$ the whole line $r(x, y, z), r \in \mathbb{R}$ through (in this example) 0. For example, if we intersect the previous cylinders with the plane $z = 1$ and take the cone with vertex at $(0, 0, 0)$ then these are described by the following (so called “homogenous”) equations: $x^2 + y^2 - z^2 = 0$, $x^2 - y^2 - z^2 = 0$, $x^2 - yz = 0$.
And finally we have the quadratic surfaces which have neither singular points nor are they cylinders.
The ellipsoids $x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0$,
1-sheeted hyperboloids $x^2/a^2 + y^2/b^2 - z^2/c^2 - 1 = 0$,
2-sheeted hyperboloids $x^2/a^2 + y^2/b^2 - z^2/c^2 + 1 = 0$,
elliptic paraboloids $x^2/a^2 + y^2/b^2 - z = 0$ and
hyperbolic paraboloids $x^2/a^2 - y^2/b^2 - z = 0$.

All other quadratic surfaces are obtained via coordinate transformations from these examples. Try the
Experiment: Select Implicit from the Surface Menu and type any quadratic equation into UserDefined. Compare the displayed surface with those described above. (Note that there may be no solutions.)

The 1-sheeted hyperboloids and the hyperbolic paraboloids have an unexpected special property, they carry two families of straight lines. The hyperbolic paraboloid $x^2 - y^2 - z = 0$ is cut by the parallel family $x + y = const$ of planes in (disjoint) lines and also by the parallel planes $x - y = const$. The 1-sheeted hyperboloid $x^2 + y^2 - z^2 - 1 = 0$ is a surface of revolution. Its tangent plane $x = 1$ intersects it in the pair of orthogonal lines $(y + z)(y - z) = 0,\ x = 1$. Rotation around the z-axis gives two families of lines on the surface. Each tangent plane cuts the surface in two lines, one from each family.

Explicit parametrizations

Ellipsoid:
$x = aa \cdot \sin\theta \cos\varphi,\ y = bb \cdot \sin\theta \sin\varphi,\ z = cc \cdot \cos\theta,$
1-sheeted hyperboloid:
$x = aa \cosh u \cos\varphi,\ y = bb \cosh u \sin\varphi,\ z = cc \sinh u,$
2-sheeted hyperboloid (2nd sheet $z \to -z$):
$x = aa \sinh u \cos\varphi,\ y = bb \sinh u \sin\varphi,\ z = cc \cosh u,$
Elliptic paraboloid:
$x = aa \cdot u \cos\varphi,\ y = bb \cdot u \sin\varphi,\ z = cc \cdot u^2,$
Hyperbolic paraboloid:
$x = aa \cdot u,\ y = bb \cdot v,\ z = cc \cdot uv.$

H.K.