Implicit Surfaces or

Level Sets of Functions

Surfaces in $\mathbb{R}^3$ are usually described either as "parametrized images" $F : D^2 \to \mathbb{R}^3$ or else as "implicit surfaces", i.e., as a *level set* of a function $f : \mathbb{R}^3 \to \mathbb{R}$, (the set of points $f^{-1}(v)$ where f has a fixed value v).

One can more easily make images of parametrized surfaces than of implicit surfaces, because every point $p \in D$ can be mapped with the given function F to obtain 'explicitly' a point $F(p)$ of the surface. Note however that the opposite problem: "Given a point in $\mathbb{R}^3$, decide whether it lies on the surface" does not have an easy answer. For an implicit surface, on the other hand, it is easy to decide whether a given point in $\mathbb{R}^3$ is on the surface (simply check $f(x) = v$), but no point is given explicitly, one has to use some algorithm to find points $x \in \mathbb{R}^3$ which satisfy $f(x) = v$.

And even after one has found many points on the surface, how does one connect them, what is a good way to represent the surface? The *raytracing* method has been invented as one solution. Choose some center point C, think of it to be near the eyes of the viewer. Connect each pixel of the screen with C by a line and decide whether this line meets the surface. Of all the intersection points on the line choose the one closest to C, compute the normal of the surface at this point x (i.e. compute $\mathrm{grad} f(x)$) and decide with this information what color light (from fixed colored light sources) will be reflected by the surface at x towards C, and color that pixel accordingly. In this way one produces an image which presents the surface as if it were an illuminated object. The computation used to take very long, but with todays computers such pictures can be computed while you wait (although not quite fast enough for real-time rotations). These pictures look very realistic, but of course they show only what is visible from the viewer.

A second method is offered by 3D-XplorMath. Imagine that the surface is intersected with random lines until around 10,000 points have been found on the implicit surface. Then red-green stereo is used to project these points to the screen. When viewed through stereo glasses one sees all these points in their correct positions in space, and our brain interpolates the picture and lets us see the surface in space. In this representation all parts of the surface (within some viewing sphere) can be seen—not just the front-most portions. Since it is possible to achieve a fairly uniform distribution of points on level surfaces, one sees many points in the direction towards contours of the surface. This emphasis of the contour points is so strong that one also gets a fair impression of the surface even if one does not look through red-green glases. Moreover, this method is fast enough for real time rotations.

What Surfaces Can One See?

Of all the closed surfaces only spheres and tori have explicit parametrizations. Therefore pictures of parametrized surfaces will show only portions of a complicated surface ("patches") or they will show spheres and tori. By contrast, even fairly simple functions can have level surfaces that are more complicated than tori—the so-called "bretzel" surfaces of genus $g > 1$.

How can one find such functions? As an example, consider two circles of radius r, in the x-y-plane, with midpoints ± 1 on the x-axis. This set is described as the intersection of the plane $g(x, y, z) := z = 0$ with the zero set of the function

$$h(x, y, z) := ((x-1)^2 + y^2 - r^2) \cdot ((x+1)^2 + y^2 - r^2).$$

Now define

$$f(x, y, z) := h(x, y, z)^2 + g(x, y, z)^2.$$

Clearly, the zero set of f is the union of the two circles, which is *not* a surface, because $\operatorname{grad} f$ vanishes along this zero set. However, most of the levels $\{(x, y, z); f(x, y, z) = v > 0\}$ are surfaces without singularities. If the two circles intersect ($r > 1$), then for very small v the levels are the boundary of a thickening of the two circles, i.e., surfaces of genus 3. As v increases either the middle hole or the two outside holes close first (depending on r). For large v the level surfaces are (not completely round) spheres. Each time such a topological change occurs we observe one special surface, it is not smooth like the other levels, but has one or more cone like singularities.
If $r < 1$ then for very small v the levels are separate tori. Then either the tori grow together to a genus 2 surface, or the holes of the tori close first and later the two sphere like surfaces grow together.

Other Functions Supplied by 3D-XplorMath

Note: One should always experiment with the level value v of the function f. For small values of v one will see how the function was designed. Most default morphs vary $v = ff$.

Pretzel.
Genus 3. Tube about a Figure 8:

$$f(x,y,z) := (((x-1)^2+y^2-aa^2)\cdot((x+1)^2+y^2-aa^2))^2 + z^2.$$

Bretzel2.
Genus 2. Tube about a Figure 8:

$$f(x,y,z) := ((1-x^2)x^2 - y^2)^2 + z^2.$$

Bretzel5.
Genus 5. Tube around two intersecting ellipses:

$$f(x,y,z) := ((x^2 + y^2/4 - 1)\cdot(x^2/4 + y^2 - 1))^2 + z^2.$$

Pilz.
Genus 3. Tube about circle and orthogonal ellipse:

$$f(x,y,z) := (\,(x^2 + y^2 - 1)^2 + (z-1)^2) \\ (x^2/aa^2 + (z-hh)^2 - 1)^2 + y^2).$$

OrthoCircles.

Genus 5. Tube about 3 orthogonal circles:

$$\begin{aligned} f(x,y,z) :=& ((x^2+y^2-1)^2+z^2)\cdot \\ & ((y^2+z^2-1)^2+x^2)\cdot \\ & ((z^2+x^2-1)^2+y^2). \end{aligned}$$

DecoCube.

Genus 5,13,17. Tube about circles on cube faces:

$$\begin{aligned} f(x,y,z) :=& ((x^2+y^2-cc^2)^2+(z^2-1)^2)\cdot \\ & ((y^2+z^2-cc^2)^2+(x^2-1)^2)\cdot \\ & ((z^2+x^2-cc^2)^2+(y^2-1)^2). \end{aligned}$$

H.K.