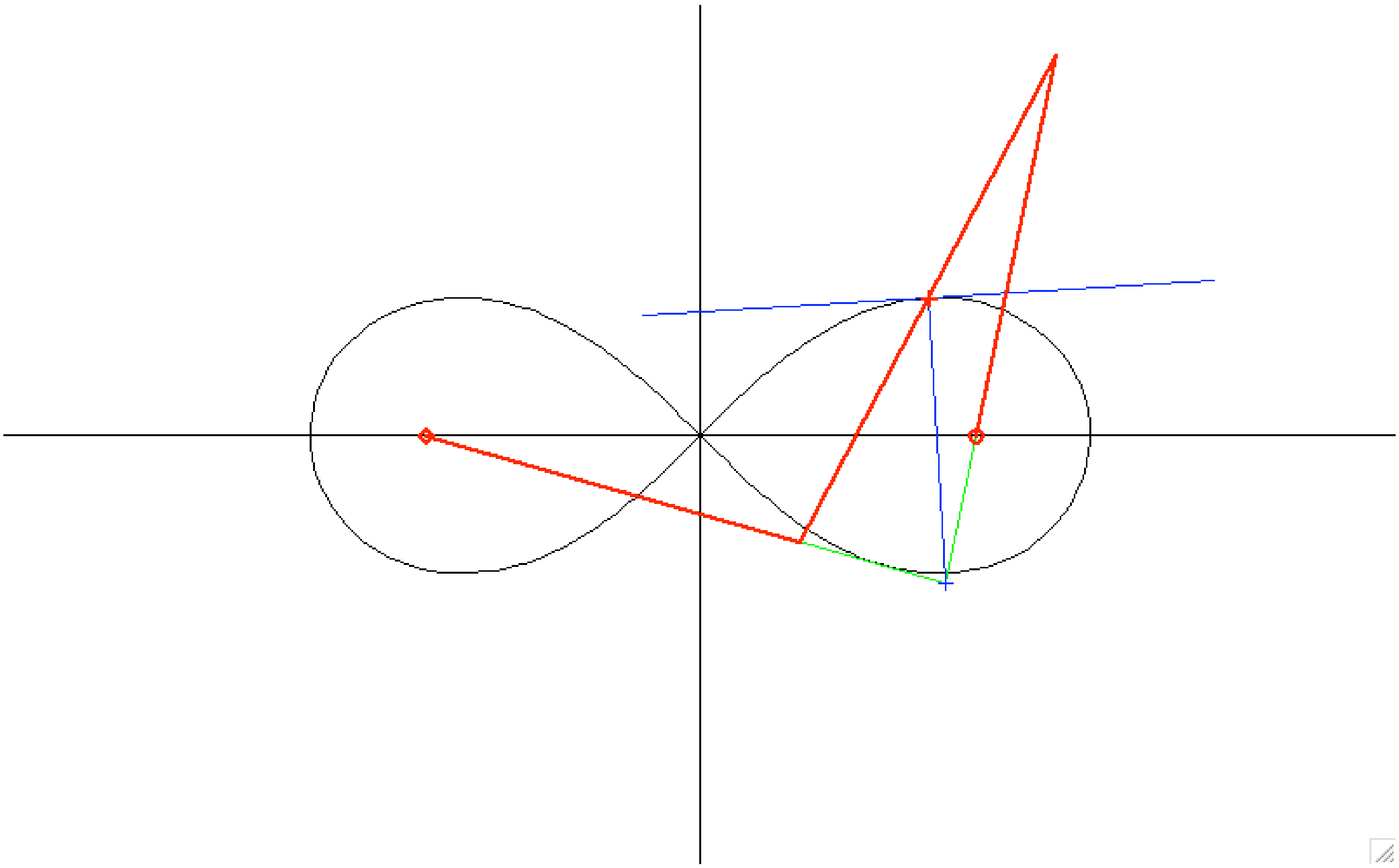
Lemniscate



The Lemniscate is a figure-eight curve with a simple **mechanical construction** attributed to Bernoulli: Choose two 'focal' points F_1, F_2 at distance L, then take three rods, one of length L, two of length $L/\sqrt{2}$. The short ones can rotate around the focal points and they are connected by the long one with joints which allow rotation. This machine has one degree of freedom and the midpoint of the long rod traces out the Lemniscate when one of the short rods is rotated.

Mechanical constructions of curves often come with simple **tangent constructions**. We imagine that a plane is attached to the long rod. Then every point of this plane traces out a curve when the rods move. At each moment the endpoints of the long rod move orthogonal to the short rods (namely on circles around the focal points). This says that the straight extensions of the short rods (green) intersect in the momentary center of rotation. At this moment every point of the plane rotates around this center so that the tangent of each point's orbit is orthogonal to its connection with the momentary center of rotation. (Compare the other mechanically constructed curves.)

The Lemniscate has the implicit equation:

$$(x^2 + y^2)^2 = x^2 - y^2.$$

Divide this by $r^2 := x^2 + y^2$ to get the polar form:

$$r^2 = \cos(\phi)^2 - \sin(\phi)^2.$$

Parametrizations are not unique, here is one:

$$x(t) := \cos(t)/(1 + \sin(t)^2)$$

$$y(t) := \sin(t)\ \cos(t)/(1 + \sin(t)^2).$$

The points $F_1, F_2 := \pm 1/\sqrt{2}$ are called Focal points of the Lemniscate because of the special property: $|P - F_1| \cdot |P - F_2| = |F_1 - F_2|^2/4$.

If one takes the complex square root of a circle which touches the y-axis from the right at 0 then one also obtains a Lemniscate. In the Conformal Category, choose $z \to \sqrt{z}$, and then in the Action Menu, select Choose Circle by Mouse, and create a circle that is tangent to the y-axis at 0.)

The inversion map: $(x, y) \mapsto (x, y)/(x^2 + y^2)$ often transforms some interesting curve into another interesting curve. And indeed, the Lemniscate, with the above parametrization, is transformed by inversion into the curve $x = 1/\cos(t),\ y = \sin(t)/\cos(t)$. Since $x^2 - y^2 = 1$, this is a hyperbola, so we could also have obtained the Lemniscate from the standard hyperbola by inversion.

We note that not every figure-eight curve is a Lemniscate, another figure-eight is obtained by the simpler parametrization:

$$x(t) := \cos(t)$$
$$y(t) := \sin(t) \cdot \cos(t),$$

which has the implicit equation $y^2 = x^2(1 - x^2)$.