The Circle

Except for the straight line, the circle is perhaps the simplest and best known of all plane curves. In addition, it is a special case of the ellipse. So why do we include it in our list of special plane curves? One reason of course is that, because it is so well-known, any list of plane curve that omits it seems somehow deficient. But there is another good reason— many of the curves that have individual names were already considered (and named) by the ancient Greeks, and a large class of these can be obtained by rolling one circle on the inside or the outside of some other circle. The Greeks were interested in rolling constructions because it was their main tool for describing the motions of the planets (Ptolemy). The following curves from the Plane Curve menu can be obtained by rolling constructions:

Cycloid, Ellipse, Astroid, Cardioid, Limaçon, Nephroid.

Not all geometric properties of these curves follow easily from their definition as rolling curve, but in some cases the connection with complex functions (Conformal Category) does.

Cycloids arise by rolling a circle on a straight line. The parametric equations code for such a cycloid is

P.x := aa*t - bb*sin(t)
P.y := aa - bb*sin(t) , aa = bb

Cycloids have other cycloids of the same size as evolute (Action Menu: "Show Osculating Circles with Normals"). This fact is responsible for Huyghen's cycloid pendulum to have a period independent of the amplitude of the oscillation.

Ellipses are obtained if *inside* a circle of radius aa another circle of radius r = hh = 0.5*aa rolls and then traces a curve with a radial stick of length R = ii*r. The parametric equations for such an ellipse is

P.x := (R+r)*cos(t)
P.y := (R-r)*sin(t)

In the visualization of the complex map $z \to z + 1/z$ in Polar Coordinates the image of the circle of Radius R is such an ellipse with $r = 1/R$.

Astroids are obtained if *inside* a circle of radius aa another circle of radius r = hh = 0.25*aa rolls and then traces a curve with a radial stick of length R = ii*r = r. Parametric equations for such Astroids are

P.x := (aa-r)*cos(t) + R*cos(4*t)
P.y := (aa-r)*sin(t) - R*sin(4*t)

Astroids can also be obtained by rolling the *larger* circle of radius r = hh = 0.75*aa (put gg = 0 in this case). Another geometric construction of the Astroids uses the fact that the length of the segment of each tangent between the x-axis and the y-axis has **constant** length.

Cardioids and Limaçons are obtained if *outside* a circle of radius aa another circle of radius $r = hh = -aa$ rolls and then traces a curve with a radial stick of length R = ii*r, ii = 1 for the Cardioids, ii > for the Limaçons:

Parametric equations for Cardioids and Limaçons are

P.x := (aa+r)*cos(t) + R*cos(2*t)
P.y := (aa+r)*sin(t) + R*sin(2*t)

The Cardioids and Limaçons can also be obtained by rolling the larger circle of radius r = hh = + 2*aa; now ii < 1 for the Limaçons. Note that the fixed circle is *inside* the larger rolling circle.
The evolute of the Cardioid (Action Menu: "Show Osculating Circles with Normals") is a smaller Cardioid. The image of the unit circle unter the complex map $z \to w = (z^2 + 2z)$ is a Cardioid; images of larger circles are Limaçons. Inverses $z \to 1/w(z)$ of Limaçons are figure-eight shaped, one of them is a Lemniscate.

Nephroids are generated by rolling a circle of one radius outside of a second circle of twice the radius, as the program demonstrates. With $R = 3r$ we thus have the parametrization
P.x := R*cos(t) + r*cos(3*t)
P.y := R*sin(t) + r*sin(3*t)
As with Cardioids and Limaçons one can also make the radius for the drawing stick shorter or longer: Choose in the Menu "Circle" and set the parameters $aa = 1, hh = -0.5, ii = 1$ for the Nephroid and $ii > 1$ for its looping relatives.

The complex map $z \to z^3 + 3z$ maps the unit circle to such a Nephriod. To see this, in the Conformal Map Category, select $z \to z^{ee} + ee \cdot z$ from the Conformal Map Menu, then choose Set Parameters from the Settings Menu and put $ee = 3$.

Back in the Plane Curves Category, select Nephroid and then in the Action Menu: "Show Osculating Circles with Normals". The Normals envelope a smaller Nephroid.

H.K.