Cassinian Oval *

The Cassinian Ovals (or Ovals of Cassini) were first studied in 1680 by Giovanni Domenico Cassini (1625–1712, aka Jean-Dominique Cassini) as a model for the orbit of the Sun around the Earth.

A Cassinian Oval is any plane curve that is the locus of all points P such that the product of the distances of P from two fixed points has some fixed value, that is: $\overline{P\,F_1}\,\overline{P\,F_2} = c$, where F_1 and F_2 are two points and c is a constant. Note the analogy with the definition of an ellipse (where product is replaced by sum). As with the ellipse, the two points F_1 and F_2 are called the *foci* of the oval. If we locate the origin of our coordinates at the midpoint of the two foci and choose for the x-axis the line joining them, then the foci will have the coordinates $(a, 0)$ and $(-a, 0)$. Following convention we will let b denote the square root of c, and then the condition for a point $P = (x, y)$ to lie on the oval becomes:

$$\sqrt{(x-a)^2 + y^2}\sqrt{(x+a)^2 + y^2} = b^2$$

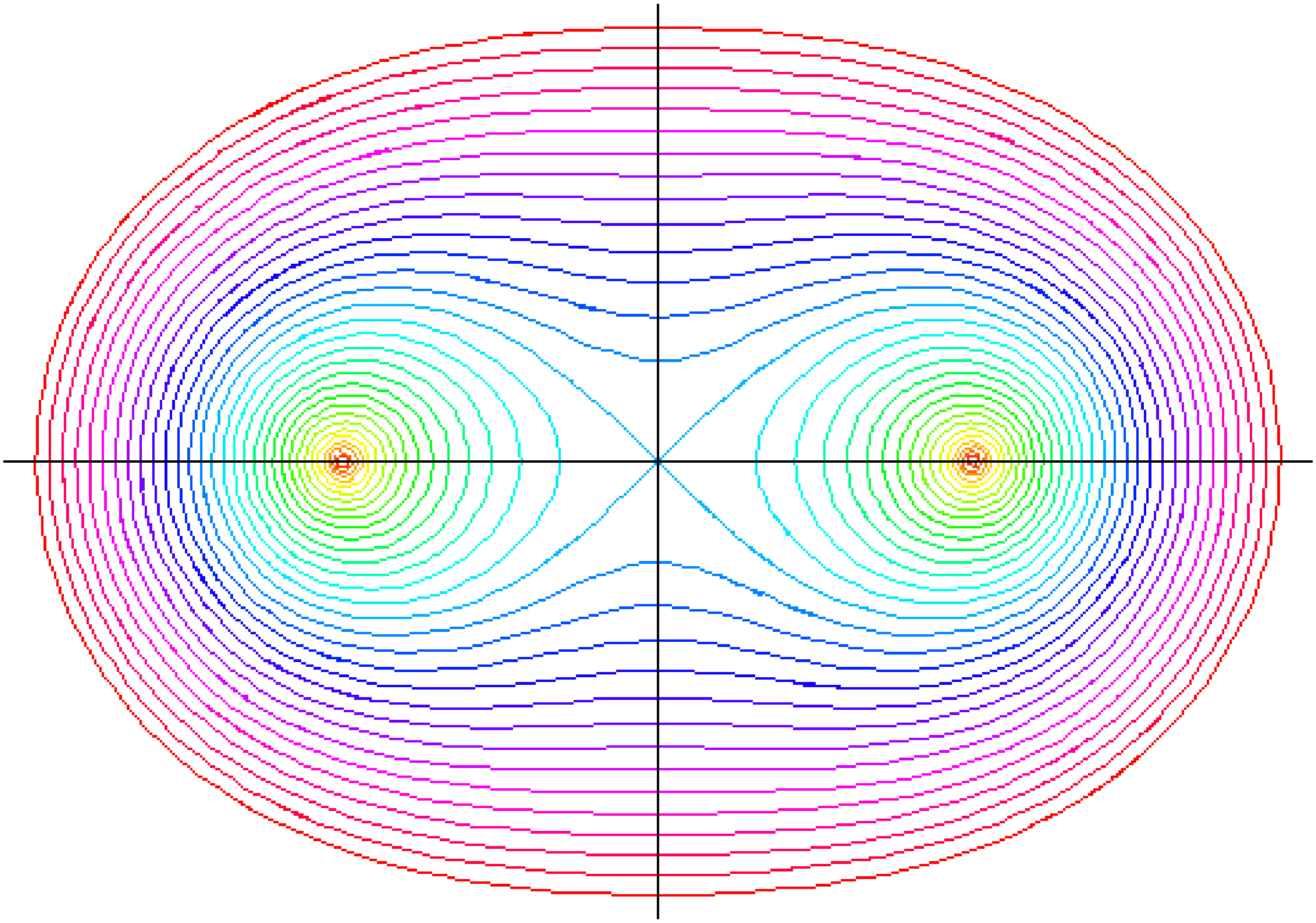
and on squaring the two sides, we end up with the following quartic polynomial equation for the Cassinian Oval:

$$((x-a)^2 + y^2)((x+a)^2 + y^2) = b^4$$

When b is less that half the distance $2a$ between the foci, i.e., $b/a < 1$, there are two branches of the curve. When $a = b$, the curve has the shape of a figure eight and is known as the Lemniscate of Bernoulli.

*This file is from the 3D-XploreMath project.
Please see http://rsp.math.brandeis.edu/3D-XplorMath/index.html

The following image shows a family of Cassinian Ovals with $a = 1$ and several different values of b.



In 3D-XploreMath, you can change the value of parameter b in the menu Settings → SetParameters. An animation of varying values of b can be seen from the menu Animate→ Color Morph.

Bipolar equation: $r_1 r_2 = b^2$

Polar equation: $r^4 + a^4 - 2r^2 a^2 \cos(2\theta) = b^4$

The parametric formula for Cassinian oval is $\sqrt{M/2}(\cos(t), \sin(t))$, where M is

$$2a^2 \cos(2t) + 2\sqrt{(-a^4 + b^4) + a^4(\cos(2t))^2},$$

with $0 < t \leq 2\pi$, and $a < b$. This parametrization only generates parts of the curve when $a > b$.

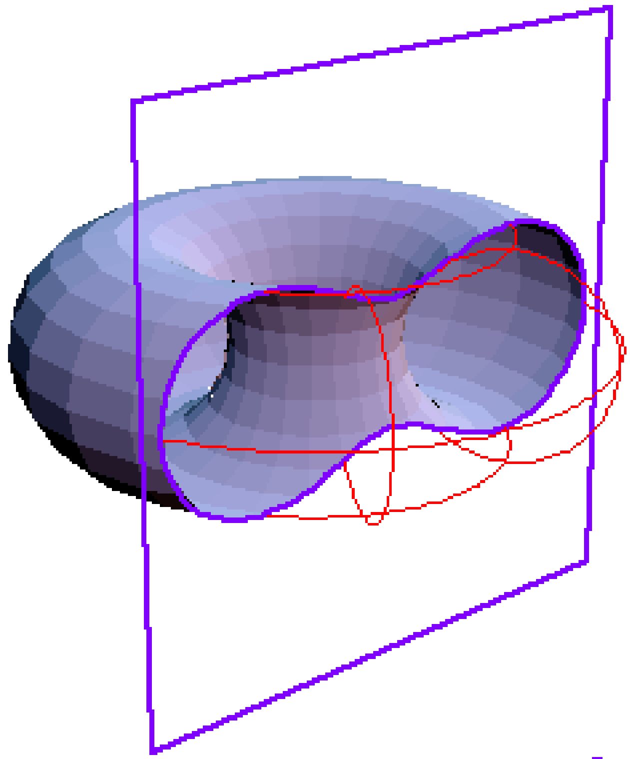
Cassinian Ovals as sections of a Torus

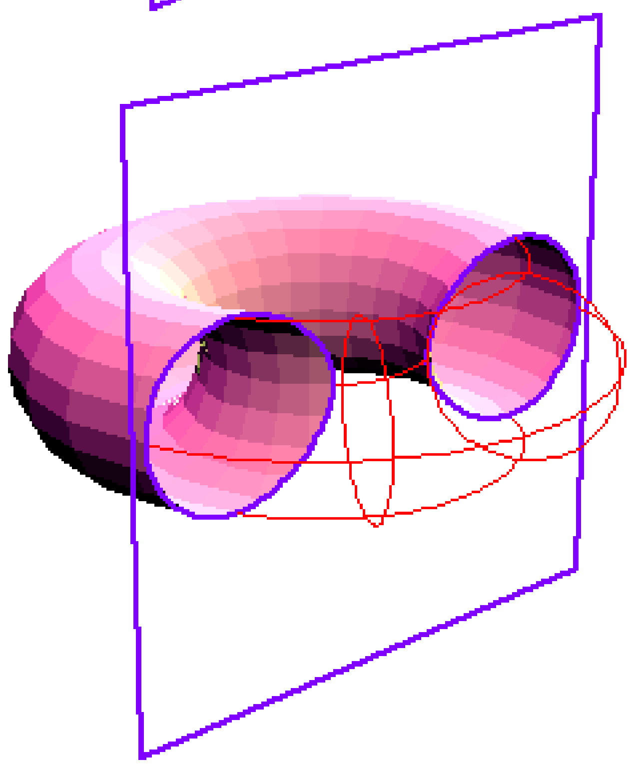
Let c be the radius of the generating circle and d the distance from the center of the tube to the directrix of the torus. The intersection of a plane c distant from the torus' directrix is a Cassinian oval, with $a = d$ and $b^2 = \sqrt{4}cd$, where a is half of the distance between foci, and b^2 is the constant product of distances.

Cassinian ovals with a large value of b^2 approch a circle, and the corresponding torus is one such that the tube radius is larger than the center to directrix, that is, a self-intersecting torus without the hole. This surface also approaches a sphere.

Note that the two tori in the figure below are not identical.

Arbitrary vertical slices of a torus are called Spiric Sections. In general they are *not* Cassinian ovals.





Proof outline: Start with the equation of a torus $(\sqrt{x^2+y^2}-d)^2+z^2=c^2$. Eliminate the square root and regroup to one side. Substitute $d=a$ and $c=b^2/(\sqrt{4}*a)$. Now do the same with Cassinian oval implicit equation $\sqrt{(x-a)^2+y^2}\sqrt{(x+a)^2+y^2} = b^2$. One sees that, up to a scale-factor and a rotation, the two equations match.

Curves that are the locus of points the product of whose distances from n points is constant are discussed on pages 60–63 of Visual Complex Analysis by Tristan Needham.

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