The Spherical Helicoids

See also: K=1 Surfaces of Revolution

A general helicoid can be obtained by applying a screw motion to a planar curve. Suppose the curve is given as $s \mapsto (f(s), g(s))$, then the helicoid can be written as

$$F(s,t) = (f(s)\cos t, f(s)\sin t, g(s) + ht)$$

with a non-zero constant h. The surface which is generally called *the* helicoid arises as the special case $f(s) = s$ and $g(s) = 0$. Our more general class of helicoids is useful because it allows to construct concrete examples of surfaces that are otherwise elusive.

For example, let us specialize a little by taking $f(s) = s$ (so that the curve is a graph over the t-axis). Then the Gauss curvature of these helicoids is

$$K = \frac{-h^2 + s^3\, g'(s)\, g''(s)}{\left(h^2 + s^2 + s^2\, g'(s)^2\right)^2}$$

This is just a first order ODE for $g'(s)$, and it is easy to check that

$$g'(s) = \sqrt{-1 - \frac{h^2}{s^2} + \frac{1}{a - \sqrt{K}\, s^2}}$$

is a 1-parameter family of solutions in the case that K is a (positive) constant. The family parameter is a. Observe that the additional integration constant we get by integrating g' only amounts to a vertical translation of the surface.

As all *complete* surfaces of positive constant curvatures are round spheres, any other example must necessarily develop singularities. This was for a long time a perfectly good reason to ignore them, and now it has been for a while a perfectly good reason to find them interesting.

For our spherical helicoids, the singularity arises as the curve $s = s_0$ for the value of s_0 where the integrand $g'(s)$ becomes 0, and is thus a horizontal circle.

The surface appears as an exercise in Eisenhart's *A Treatise On The Differential Geometry of Curves And Surfaces*. It was certainly well known much earlier.

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An alternative description

The velocity vector field X of a screw motion in $\mathbb{R}^3$ is $X(x,y,z) = (-y, x, h)$. On a surface that is screw motion invariant one finds that the unit speed curves γ orthogonal to X are geodesics: The covariant derivative $\frac{D}{ds}\gamma'$ is zero because it is orthogonal to γ' and to $X \circ \gamma$, namely:
$0 = \langle \gamma', \gamma' \rangle' = 2\langle \gamma', \frac{D}{ds}\gamma' \rangle$
$0 = \langle \gamma', X \circ \gamma \rangle \Longrightarrow 0 = \langle \frac{D}{ds}\gamma', X \circ \gamma \rangle + 0.$
A Killing field restricts along a geodesic to a Jacobi field $J = X \circ \gamma$, and on a 2-dim surface we have (because of $\gamma' \perp J$) that $J/|J|$ is a parallel field. Therefore we get from the Jacobi equation

$$|J|'' = -K \cdot |J|, \text{ i.e., if } K = 1 \text{ then } |J(s)| = a \cdot \cos(s).$$

We write $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ and abbreviate $r^2 = x^2 + y^2,\ r(s) := \sqrt{\gamma_1^2(s) + \gamma_2^2(s)}$. This gives:

$$|J(s)| = \sqrt{r^2(s) + h^2} = a \cdot \cos(s).$$

What remains is a first order ODE for $\gamma(s)$. We abbreviate the radial horizontal vector field as n_r, i.e.,

$n_r \circ \gamma = (\gamma_1, \gamma_2, 0)/r$, and we extend the orthonormal vectors $\{n_r, X/|X|\}$ to an orthonormal basis with $n_z(x,y,z) := (y \cdot h/r, -x \cdot h/r, r)/\sqrt{r^2+h^2}$. Then

$$\gamma'(s) = r'(s) \cdot n_r \circ \gamma(s) + \sqrt{1 - r'(s)^2} \cdot n_z \circ \gamma(s).$$

One needs $|r'(s)| \leq 1$ and this condition gives

$$s_{max} := \sqrt{\frac{h^2}{a^4} + \left(\frac{1-a^2}{2a^2}\right)^2} + \frac{a^2-1}{2a^2}.$$

One can check that this condition implies

$$r^2(s) = a^2 \cos^2(s) - h^2 \geq 0.$$

The final family, with parameters a and h, is

$$F(s,t) = \begin{pmatrix} \gamma_1(s)\cos t - \gamma_2(s)\sin t \\ \gamma_1(s)\sin t + \gamma_2(s)\cos t \\ \gamma_3(s) + h \cdot t \end{pmatrix}.$$

If $h = 0$ one obtains surfaces of revolution with meridian $\gamma(s)$, and $a = 1$ is the sphere.