Ruled Surfaces *

Cylinders, Cones, 1-sheeted Hyperboloid, Hyperbolic Paraboloid, Helicoid, Right Conoid, Whitney Umbrella. *In other sections:* Double Helix, Möbius Strip.

Informally speaking, a **ruled surface** is one that is a union of straight lines (the rulings). To be more precise, it is a surface that can be represented parametrically in the form:

$$x(u, v) = \delta(u) + v * \lambda(u)$$

where δ is a regular space curve (i.e., δ' never vanishes) called the **directrix** and λ is a smooth curve that does not pass through the origin. Without loss of generality, we can assume that $|\lambda(t)| = 1$. For each fixed u we get a line $v \mapsto \delta(u) + v * \lambda(u)$ lying in the surface, and these are the rulings. (Some surfaces can be parameterized in the above form in two essentially different ways, and such surfaces are called **doubly-ruled surfaces**.)

A ruled surface is called a **cylinder** if the directrix lies in a plane P and $\lambda(u)$ is a constant direction not parallel to P, and it is called a **cone** if all the rulings pass through a fixed point V (the vertex).

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

Two more interesting examples are quadratic surfaces:

The **Hyperboloid of One Sheet:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

which is in fact doubly-ruled, since it can be given parametrically by:

$$x^+(u,v) = a(\cos(u) - v\sin(u)), b(\sin(u) + v\cos(u), cv)$$

and

$$x^-(u,v) = a(\cos(u) + v\sin(u)), b(\sin(u) - v\cos(u), -cv),$$

and the **Hyperbolic Paraboloid:**

$$(x,y,z) = (a\,u, b\,v, c\,u\,v) = a(u,0,0) + v(0,b,c\,u).$$

Another interesting ruled surface is a *minimal surface*:

the **Helicoid**, $aa = 0$, (Catenoid, $aa = \pi/2$) in the family:

$$\begin{aligned}F(u,v) = bb\sin(aa)\big(&\cosh(v)\cos(u),\ \cosh(v)\sin(u),\ v\big)\\ &+ bb\cos(aa)\big(\sinh(v)\sin(u),\ -\sinh(v)\cos(u),\ u\big)\\ = &\ \sin(aa)\big((0,0,bb\,v) + bb\cosh(v)(\cos(u),\sin(u),0)\big)\\ &+ \cos(aa)\big((0,0,bb\,u) + bb\sinh(v)(\sin(u),-\cos(u),0)\big).\end{aligned}$$

A ruled surface is called a (generalized) **right conoid** if its rulings are parallel to some plane, P, and all pass through a line L that is orthogonal to P. *The* **Right Conoid** is given by taking P to be the xy-plane and L the z-axis:

Parametrized: $F(u,v) = (v\cos u, v\sin u, 2\sin u)$,

Implicitly: $(\frac{x}{y})^2 - \frac{4}{z^2} = 1.$

This surface has at $(\sin(u) = \pm 1, v = 0)$ two pinch point singularities. The `default morph` in 3DXM

deforms the **Right Conoid** to a **Helicoid**

so that the two stable pinch point singularities disappear, at the final moment, through two unstable singularities:

$$F_{aa}(u,v) = (\, v\cos(u), v\sin(u), 2aa\sin(u) + (1-aa)u\,)\,.$$

Famous for such a singularity is the **Whitney Umbrella**, another right conoid with rulings parallel to the x-y-plane:

$$F(u,v) = (u\cdot v, u, v\cdot v), \;\; \textit{implicitly: } x^2 - y^2 z = 0.$$

Again the `default morph` emphasizes the visualization of the singularity by embedding the **Whitney Umbrella** into a family of ruled surfaces, which develop a second pinch point singularity that closes the surface at the top:

$$F_{aa}(u,v) = \begin{pmatrix} u\cdot(aa\cdot v + (1-aa)\sin(\pi v)) \\ u \\ aa\cdot v^2 - (1-aa)\cos(\pi v) \end{pmatrix}.$$

R.S.P.