About Double Enneper

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The surfaces Wavy Enneper, Catenoid Enneper, Planar Enneper and Double Enneper are finite total curvature minimal immersions of the once or twice punctured sphere—shown with standard polar coordinates. These surfaces illustrate how the different types of ends can be combined in a simple way.

The pure Enneper surfaces $(\text{Gauss}(z) = z^k,\ dh = \text{Gauss}(z)\ dz)$ and the Planar Enneper surfaces

$$(\text{Gauss}(z) = z^k,\ dh = \text{Gauss}(z)/z^2\ dz)$$

have been re-discovered many times, because the members of the associate family are *congruent* surfaces (as can be seen in an associate family morphing) and the Weierstrass integrals integrate to polynomial (respectively) rational immersions. Double Enneper was one of the early examples in which I joined two classical surfaces by a handle. The Weierstrass data are:

Gauss map : $Gauss(z) = z^{ee-1}(z^{ee} - A^{ee})/(A^{ee}z^{ee} - 1)$
Differential: $dh = e^{i\varphi}(1 - (z^{ee} - z^{-ee})/(A^{ee} - A^{-ee}))\, dz/z$.
with $A = \sqrt{aa} \cdot \exp(i\alpha), \alpha = \pi bb/ee$
and $(A^{ee}+A^{-ee}) \tan\varphi = -(A^{ee}-A^{-ee}) \tan 2\alpha$. The last equality is needed to avoid periods of the Weierstrass integral.

In this example the parameter aa controls the size of the neck between the top and bottom Enneper ends (it should be kept in the range $3 < aa < 7$). The parameter bb rotates the top and the bottom ends relative to each other, The integer parameter $ee = 2, 3,...$ determines the winding number of the Enneper ends and also the rotational symmetry of the surface—the default is

$ee = 2$. And umin and umax control how far into the ends one computes.

Try a *Cyclic Morph* with the default parameters (this rotates the upper and lower ends in opposite directions). We also suggest morphing the size (aa) of the handles. If the ends intersect too much (e.g. for too large u-range or too large ee) one has to reduce the u-range. The Cyclic Morph is also interesting for large ee, for example $ee = 12, umin = -1.45, umax = 1.5$ and reduced scaling.

Formulas are taken from:

H. Karcher, Construction of minimal surfaces, in "Surveys in Geometry", Univ. of Tokyo, 1989, and Lecture Notes No. 12, SFB 256, Bonn, 1989, pp. 1–96.

For a discussion of techniques for creating minimal surfaces with various qualitative features by appropriate choices of Weierstrass data, see either [KWH], or pages 192–217 of [DHKW].

[KWH] H. Karcher, F. Wei, and D. Hoffman, The genus one helicoid, and the minimal surfaces that led to its discovery, in "Global Analysis in Modern Mathematics, A Symposium in Honor of Richard Palais' Sixtieth Birthday", K. Uhlenbeck Editor, Publish or Perish Press, 1993

[DHKW] U. Dierkes, S. Hildebrand, A. Kuster, and O. Wohlrab, Minimal Surfaces I, Grundlehren der math. Wiss. v. 295 Springer-Verlag, 1991