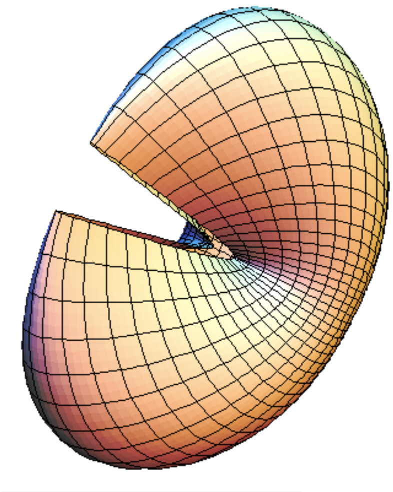
The Cross-Cap and Steiner's Roman Surface *



Opposite boundary points of this embedded disk are identified along a segment of double points when the surface is closed to make the Cross-Cap. Pinchpoint singularities form at the endpoints of the selfintersection segment.

In the 19th century images of the projective plane where found by restricting quadratic maps $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ to the unit sphere. For example a **Cross-Cap** is obtained with $f(x, y, z) = (xz, yz, (z^2 - x^2)/2$, and **Steiner's Roman Surface** with $f(x, y, z) = (xy, yz, zx)$.
Parametrizations follow by restricting to a parametrized sphere $F_{Sphere}(u, v) = \big(\sin u \cos v,\ \sin u \sin v,\ \cos u\big)$.

Steiner's surface has three self-intersection segments and six pinchpoint singularities. The `Default Morph` emphasizes a (self-intersecting) Möbius Band on this surface.

Cross-caps occur naturally as a family by a differential geometric construction. Consider at a point p of positive curvature of some surface the family of all the normal curvature circles at p. They form a cross-cap and the two

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

pinchpoint sigularities are the points opposite to p on the two principal curvature circles. The parameters aa, bb in 3DXM are the two principal curvature radii. The `Default Morph` varies bb from $bb = 0.4aa$ to $bb = aa$, a sphere. A `Range Morph` starts by taking half of each normal curvature circle and slowly extends them to full circles.

To derive a parametrization of this family of cross-caps let e_1, e_2 be a principal curvature frame at p, let κ_1, κ_2 be the principal curvatures and $r_1 := 1/\kappa_1, r_2 := 1/\kappa_2$ the principal curvature radii at p. The normal curvature in the direction $e(\varphi) := e_1 \cos\varphi + e_2 \sin\varphi$ is
$\kappa(\varphi) := \kappa_1 \cos^2\varphi + \kappa_2 \sin^2\varphi$ with $r(\varphi) := 1/\kappa(\varphi)$.
In 3DXM we parametrize the circles by $u \in [-\pi, \pi]$ and use $v = \varphi$. Denoting the surface normal by n we get the family of normal circles as

$$r(v) \cdot (-n + n \cdot \cos u + e(v) \cdot \sin u),\ u \in [0, \pi],\ v \in [0, 2\pi].$$

Finaly we take $\{e_1, e_2, n\}$ as the (x, y, z) coordinate frame and allow translation by cc along the z-axis to get our

Parametrization of the normal curvature Cross-cap

$$\begin{aligned}
r(v) &= r_1 r_2/(r_2 \cos^2 v + r_1 \sin^2 v),\\
x &= r(v) \cos v \sin u,\\
y &= r(v) \sin v \sin u,\\
z - cc &= r(v)(-1 + \cos u) = 2r(v) \sin^2(u/2),\\
u &\in [0, \pi],\ v \in [0, 2\pi].
\end{aligned}$$

To also get an *implicit equation* we observe $y/x = \tan v$ and $z^2/(x^2+y^2) = \tan^2(u/2)$.
This leads to

$$\begin{aligned} x^2/(x^2+y^2) &= \cos^2 v, \\ y^2/(x^2+y^2) &= \sin^2 v, \\ z^2/(x^2+y^2+z^2) &= \sin^2 u/2, \\ (x^2+y^2)/(x^2+y^2+z^2) &= \cos^2 u/2. \end{aligned}$$

The first two of these equations eliminate v from $r(v)$. The third one eliminates u from $z/r(v) = 2\sin^2(u/2)$ and gives (with $r_1 = aa,\ r_2 = bb$) an

Implicit equation of the normal curvature Cross-cap

$$(\frac{x^2}{aa} + \frac{y^2}{bb})(x^2+y^2+z^2) + 2z(x^2+y^2) = 0.$$

Finally, replacing z by $z - cc$ will translate the cross-cap, e.g. $cc = 2bb$ puts the pinchpoint of the smaller curvature circle to the origin of $\mathbb{R}^3$.

H.K.