# Surfaces of Constant Width *

The name of these surfaces derives from the fact that the distance between opposite parallel tangent planes is constant. See first: `Convex Curve` in the planar curve category. There the default curve and the default morph show curves of constant width and the ATO: `On Curves Given By Their Support Function` explains how they are made.

Our surfaces of constant width are also described via their

Support Function $h : \mathbb{S}^2 \mapsto \mathbb{R}$ as:

$$F(x,y,z) := h(x,y,z) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \operatorname{grad}_{\mathbb{S}^2} h(x,y,z),$$

$$\begin{aligned} h(x,y,z) &:= aa + bb\ z^3 + cc\ xy^2 + dd\ yz^2 + ee\ xz^2 \\ &\quad + ff\ xyz + gg\ xy^2z^2 > 0, \\ \text{where} \quad & x^2 + y^2 + z^2 = 1. \end{aligned}$$

The constant $aa$ has to be chosen large enough so that $h(x,y,z) > 0$. The default values are $aa = 1,\ ff = 0.66$, all others $= 0$, for tetrahedral symmetry. With just $aa, bb > 0$ one gets surfaces of revolution, with $aa, cc \neq 0$ the surfaces have 120° dihedral symmetry. Note that $gg$ is the only coefficient of a polynomial of degree 5.

Let $c(t) = (x,y,z)(t)$ be a curve on $\mathbb{S}^2$. One computes with

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

$\vec{n} := (x, y, z)^t$ that $\frac{d}{dt}F(x, y, z)(t) \perp \vec{n}$, so that $\vec{n}(x, y, z)$ is the normal of the surface parametrized by $F$. Therefore $h(x, y, z)$ is indeed the distance of the tangent plane at $F(x, y, z)$ from the origin. The origin is inside the surface because $h > 0$. All terms defining $h$, except the constant $aa$, are odd. This gives $h(x, y, z) + h(-x, -y, -z) = 2 \cdot aa$ and this is the distance between opposite tangent planes, i.e. the constant width.
Finally we compute the normal curvature, more precisely the Weingarten map $S$.

$$\frac{d}{dt}\vec{n}(t) = \dot{c}(t) =: S \cdot \frac{d}{dt}F(c(t))$$

$$\begin{aligned}\frac{d}{dt}F(x, y, z)(t) &= \langle \mathrm{grad}_{\mathbb{S}^2} h, \dot{c}(t)\rangle c(t) + h \cdot \dot{c}(t) \\ &\quad + d_{\dot{c}(t)}\mathrm{grad}_{\mathbb{S}^2} h \\ &= h \cdot \dot{c}(t) + \big(d_{\dot{c}(t)}\mathrm{grad}_{\mathbb{S}^2} h\big)^{tangential} \\ &= h \cdot \dot{c}(t) + D_{\dot{c}(t)}\mathrm{grad}_{\mathbb{S}^2} h,\end{aligned}$$

where $D_{\dot{c}}$, the tangential component of the Euclidean derivative $d_{\dot{c}(t)}$, is the covariant derivative of $\mathbb{S}^2$. We thus obtain the Weingarten map $S$ of the surface, computed in the domain $\mathbb{S}^2$ of $F$:

$$\frac{d}{dt}\vec{n}(t) = \big(h \cdot \mathrm{id} + D\mathrm{grad}_{\mathbb{S}^2} h\big)^{-1} \cdot \frac{d}{dt}F(c(t)) = S \cdot \frac{d}{dt}F(c(t)).$$

H.K.