Addition on Cubic Curves.*

See also the Action Menu of the Parabola "Show Normals through Mouse Point" and the comments in the ATO.
As an introductory example view the unit circle as a group. Then the addition of angles $\phi \in (\mathbb{R} \bmod 2\pi)$ gets translated via the parametrization

$$x = \cos(\phi), y = \sin(\phi)$$

into

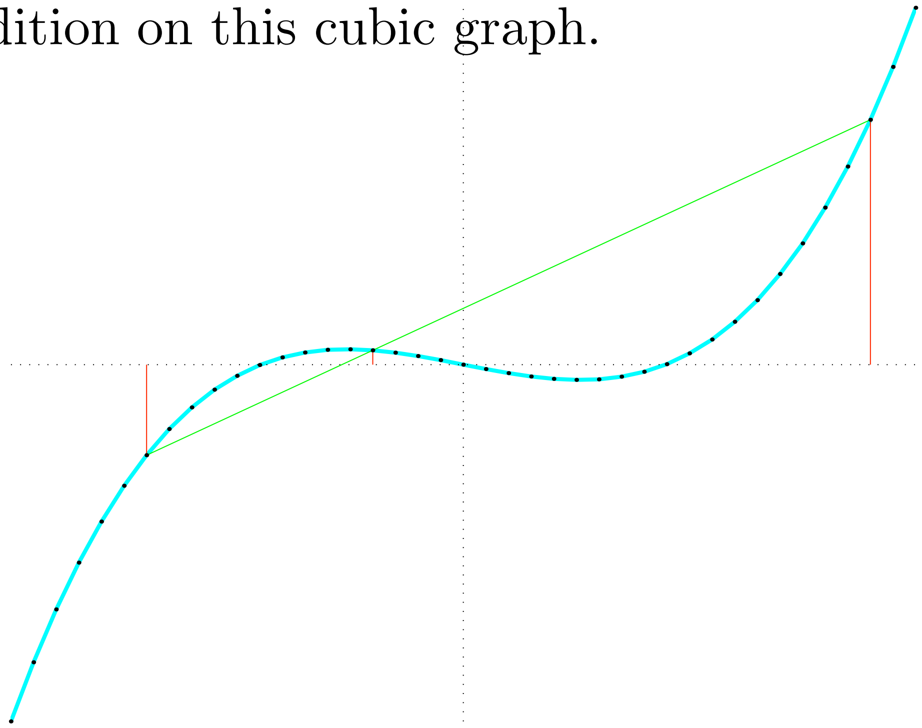
$$(x_1, y_1) \oplus (x_2, y_2) := (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1).$$

Once this addition law is known one does not need the transcendental functions sin and cos to "add" points on the circle. Even to do this addition with ruler and compass is easy. And it is amusing to note that the Pythagorean (or rational) points of the circle are a subgroup, e.g. $(3/5, 4/5) \oplus (12/13, 5/13) = ((36 - 20)/65, (15 + 48)/65)$.

In a similar way there exists a geometric addition on cubic curves, and if the cubic is parametrized with appropriate functions (defined either on $\mathbb{C}$, or on $\mathbb{C}/2\pi\mathbb{Z}$, or on $\mathbb{C}/\Gamma$, Γ a lattice in $\mathbb{C}$) then the well known addition in the domain is, under the special parametrization, the same as the geometric addition on the cubic. The simplest instance is when the cubic is the graph of a cubic polynomial without

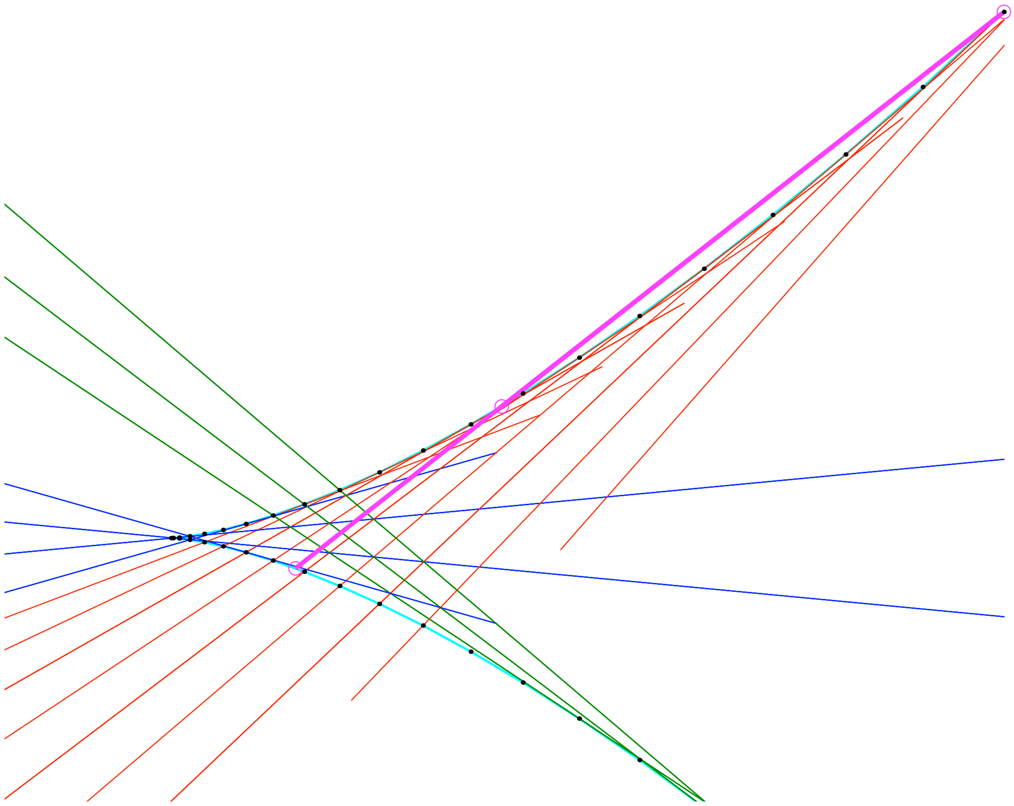
* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

quadratic term: $y = x^3 + mz + c$. Then, if we have two points $(x_1, y_1), (x_2, y_2)$ on this cubic and join them by a line, this line intersects the graph in a third point (x_3, y_3) such that $x_1+x_2+x_3 = 0$. This gives a geometric definition of addition on this cubic graph.



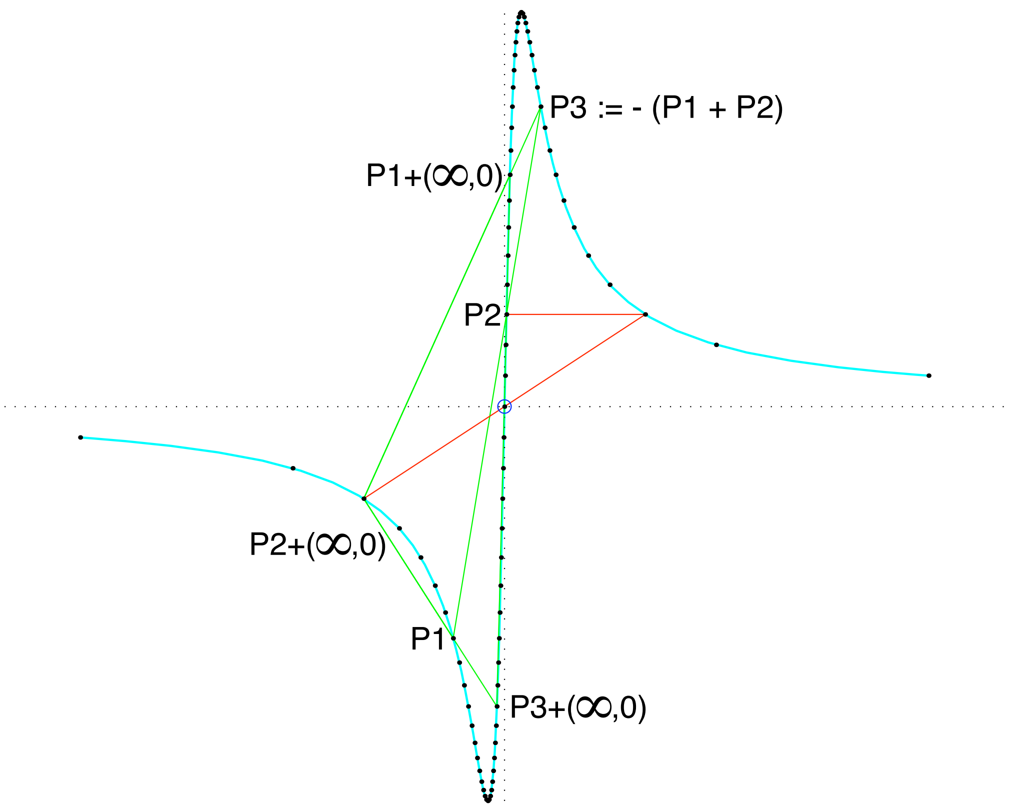
Addition on a polynomial cubic graph without quadratic term. Every line intersects so that $x_1 + x_2 + x_3 = 0$. Note discrete subgroup.

Similarly, let us map $\mathbb{C}$ bijectively onto the Cuspidal Cubic by $z \mapsto (z^2, z^3)$. In this case, if we have $z_1 + z_2 + z_3 = 0$, then the tangents at the three points (z_j^2, z_j^3) are concurrent—we have seen this as a property of the Parabola, because the Cuspidal Cubic is the evolute of the Parabola. One can also see the previous colinearity as reflecting addition, because the three points $(z_j^2, z_j^3), j = 1, 2, 3$, of this cubic lie on a line if $1/z_1 + 1/z_2 + 1/z_3 = 0$.



Addition on the cuspidal cubic $z \mapsto (z^2, z^3)$. Note the discrete subgroup. If $z_1 + z_2 + z_3 = 0$, then the tangents at these three points are concurrent. If $1/z_1 + 1/z_2 + 1/z_3 = 0$, then these three points lie on a straight line.

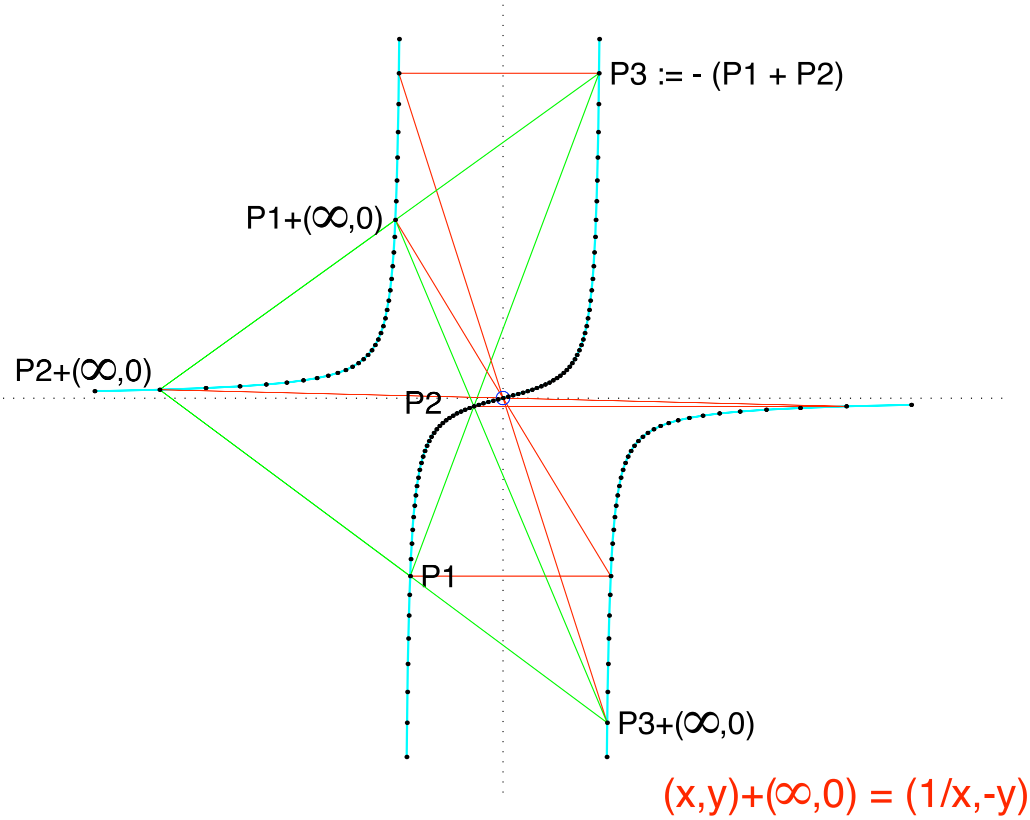
The next case is the group $\mathbb{C}/2\pi\mathbb{Z}$. The trigonometric functions identify points in $\mathbb{C}$ mod 2π. We map this group to a cubic curve by $x := \tan(z/2)$, $y := \sin(z)$, so that $y = 2x/(x^2 + 1)$ and this cubic is again a graph. The addition theorems $\tan(z + w) = (\tan(z) + \tan(w))/(1 - \tan(z)\tan(w))$ and $\sin(z+w) = \sin(z)\cos(w)+\cos(z)\sin(w)$ with $\cos(z) = 1 - 2\sin(z/2)^2 = 1 - \sin(z) \cdot \tan(z/2)$ again give an addition on this cubic graph: it is a geometric addition because the three points (x_j, y_j) lie on one line iff $z_1 + z_2 + z_3 = 0$. The name "geometric addition" is even more justified because the third point (x_3, y_3) can be constructed with ruler and compass from the other two. In fact, for repeated additions a ruler suffices: As a preparation we have to add to all points in sight the 2-division point $(\infty, 0) = (\tan(\pi/2), \sin(\pi))$ as follows: $(x, y) \oplus (\infty, 0) = (-1/x, -y)$. One needs ruler and unit circle for this. Then the lines through $(x_1, y_1), (x_2, y_2)$ and $(x_1, y_1) \oplus (\infty, 0), (x_2, y_2) \oplus (\infty, 0)$ intersect in the point $(x_3, y_3) = -(x_1, y_1) \oplus (x_2, y_2)$.



Addition group $\mathbb{S}^1$ on a cubic that is the graph of $x \mapsto y = 2x/(x^2 + 1)$, parametrized by $x := \tan(z/2)$, $y := \sin(z)$. Note the finite discrete subgroup. $(\infty, 0) = (\tan(\pi/2), \sin(\pi))$, the point at infinity, is the only point of order 2.

So far we have seen the circle part of the cylinder group $\mathbb{C}/2\pi\mathbb{Z}$. To see a generator of the cylinder we replace t, x, y

by it, ix, iy, then we obtain $x := \tanh(z/2),\ y := \sinh(z)$, so that $y = 2x/(1-x^2)$. The component of the graph through 0 is a subgroup isomorphic to $\mathbb{R}$. It represents one generator of the cylinder. The other two components represent the opposite generator with one point missing: the 2-division point opposite 0 is the point $(\infty, 0)$ on this cubic. This allows the same ruler construction of addition as before, except for a sign change in $(x,y) \oplus (\infty,0) = (+1/x, -y)$ (because $1/i = -i$).



Addition group $\mathbb{R} \cup \mathbb{R}$ on a cubic that is the graph of $x \mapsto y = 2x/(1-x^2)$ and is parametrized by $x := \tanh(z/2),\ y := \sinh(z)$. $(\infty, 0)$ is the only point of finite order. Note the infinite discrete subgroup with one finite subgroup of order 2.

Finally we come to the group $\mathbb{C}/\Gamma$. The parametrizing functions of the previous example, $\tan(z/2)$, $\sin(z)$, must be replaced by Γ-invariant, "doubly periodic" functions, also called elliptic functions. The simplest of these are those of degree two, as maps from the torus $T^2 := \mathbb{C}/\Gamma$ to the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$. Two facts are important:

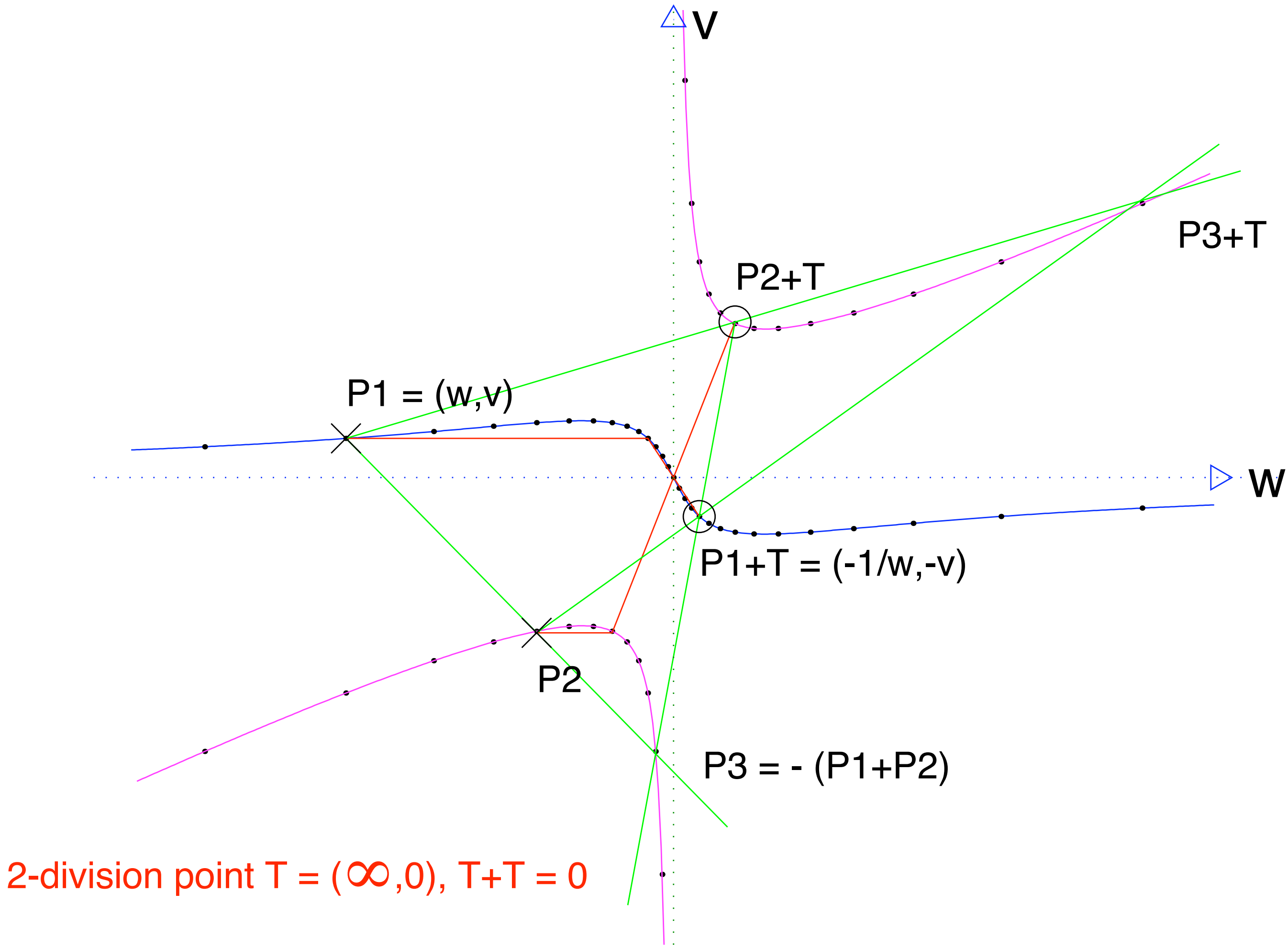
(i) Pairs of such functions satisfy cubic equations such as $(w^2+1)v = \text{const} \cdot (v^2-1)w$. The solution set of any cubic equation is called a cubic curve.

(ii) There are addition formulas, analogous to those for sin and cos.
They determine the pair $(v(z_1+z_2), w(z_1+z_2))$ from the pairs $(v(z_1), w(z_1))$ and $(v(z_2), w(z_2))$.

It turns out that these addition formulas are again "geometric" as in the previous cases, namely, the three pairs $(v(z_1), w(z_1))$, $(v(z_2), w(z_2))$, $(-v(z_1+z_2), -w(z_1+z_2))$ lie on a line. Therefore we can again define addition on the cubic geometrically:

> *Join the points to be added by a line and take the third point of intersection with the cubic as the negative of the sum.*

The addition formulas are simple enough so that the geometric addition is again a "ruler and compass construction". The compass is only needed to add 2-division points as in the previous case, all further additions can be done by intersecting lines only.



(Notice the discrete subgroup)

Addition on a general cubic:

$$(w_1, v_1) \ominus (w_2, v_2) = \left(\frac{w_1 + w_2}{1 - w_1 w_2} \cdot \frac{v_1 - v_2}{v_1 + v_2}, \frac{1 + w_1 w_2}{1 - w_1 w_2} \cdot \frac{v_1 - v_2}{1 - v_1 v_2}\right)$$

.

The elliptic functions v, w, parametrizing the above cubic curve have numerous properties that can be used to define them. For example, they are numerically accessible, since they are solutions of the following system of differential equations (compare $\tan' / \tan = 1/\cot + \cot$):

$$\frac{v'}{v} = w'(0)\left(\frac{1}{w} - w\right),$$
$$\frac{w'}{w} = v'(0)\left(\frac{1}{v} + v\right),$$

with $v'(0)/w'(0) = -2$ for the above cubic. These imply functional equations for v, w so that more similarities with the trigonometric case, like $(\sin')^2 = 1 - \sin^2$, become apparent:

$$\begin{aligned}\left(\frac{v'}{v}\right)^2 &= w'(0)^2\left(\frac{1}{w} - w\right)^2 \\ &= w'(0)^2\left(\left(\frac{1}{w} + w\right)^2 - 4\right) \\ &= v'(0)^2\left(\left(\frac{1}{v} - v\right)^2\right) - 4w'(0)^2,\end{aligned}$$

and hence: $$(v')^2 = v'(0)^2\left((1 - v^2)^2 - 4\frac{w'(0)^2}{v'(0)^2} \cdot v^2\right).$$

Every differential equation

$$(f')^2 = F(f) \text{ implies } 2f'' = F'(f).$$

The first order equation determines f' only up to sign while the second order equation determines f'' uniquely, in particular for trigonometric and elliptic functions.